HABERMAS' CONSTRUCT OF RATIONAL BEHAVIOR IN MATHEMATICS EDUCATION: NEW ADVANCES AND RESEARCH QUESTIONS

INTRODUCTION

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Habermas' construct of rational behavior deals with the complexity of discursive practices according to three interrelated elements [1]): knowledge at play (epistemic rationality); action and its goals (teleological rationality); communication and related choices (communicative rationality). Thus, it seems suitable for being applied to mathematical activities like proving and modeling that move along between epistemic validity, strategic choices and communicative requirements. The following aspects of Habermas' elaboration (1998, pp. 310-316) are relevant for us.

Concerning epistemic rationality

> In order to know something in an explicit sense, it is not, of course, sufficient merely to be familiar with facts that could be represented in true judgments. We *know* facts and have knowledge of them only when simultaneously know why the corresponding judgments are true. (...) Whoever believes that he has knowledge at his disposal assumes the possibility of a discursive vindication of corresponding true claims. (...) This does not mean, of course, that rational beliefs or convictions always consist of true judgments. (...) Someone is irrational if she puts forward her beliefs dogmatically, clinging to them although she sees that she cannot justify them. In order to qualify a belief as rational, it is sufficient that it can be held to be true on the basis of good reasons in the relevant context of justification – that is, that it can be accepted rationally. (...) The rationality of a judgment does not imply its truth but merely its justified acceptability in a given context.

These remarks concern the intentional character of rational behavior on the epistemic side and align with a view of development of knowledge (the qualifying element being the tension towards knowing "why the corresponding judgments are true"). Then a connection is established with speech and action (i.e. teleological rationality)— the latter being related to the evolutionary character of knowledge:

> Of course, the reflexive character of true judgments would not be possible if we could not represent our knowledge, that is, if we could not express it in sentences, and if we could not correct it and expand it; and this means: if we were not able also to learn from our practical dealings with a reality that resists us. To this extent, epistemic rationality is entwined with action and the use of language.

Concerning teleological rationality

> Once again, the rationality of an action is proportionate not to whether the state actually occurring in the world as a result of the action coincides with the intended state and satisfies the corresponding conditions of success, but rather to whether the actor has

> achieved this result on the basis of the deliberately selected and implemented means (or, in accurately perceived circumstances, could normally have done so).

Regarding problem solving in its widest meaning (including conjecturing, proving, modeling, finding counter-examples, generalizing, and so on), this sentence brings forth the quality of the process, which may be qualified as rational (on the teleological side) even if the original goal is not reached. The relevant feature of teleological rationality consists of the action intentionality (including the choice and use of the means to achieve the goal) and the reflective attitude towards it:

> A successful actor has acted rationally only if he (i) knows why he was successful (or why he could have realized the set goal in normal circumstances) and if (ii) this knowledge motivates the actor (at least in part) in such a way that he carries out his action for reasons that can at the same time explain its possible success.

The second condition represents a projection from the past to the future—namely, a conscious enrichment of strategies, in the case of problem solving situations.

Concerning communicative rationality

> (...) communicative rationality is expressed in the unifying force of speech oriented toward reaching understanding, which secures for the participating speakers an intersubjectively shared lifeworld, thereby securing at the same time the horizon within which everyone can refer to one and the same objective world.

From this ideal practice of communicative rationality, which creates an "intersubjectively shared lifeworld"—thus the possibility of referring to the same "objective world", Habermas moves to an evaluation of actual individual rational behavior on the communicative side:

> (...) The rationality of the use of language oriented toward reaching understanding then depends on whether the speech acts are sufficiently comprehensible and acceptable for the speaker to achieve illocutionary success with them (or for him to be able to do so in normal circumstances).

Here again the intentional, reflective character is pointed out (for the specific case of communicative rationality).

Habermas' elaboration offers a model to deal with important aspects of mathematical activity like those above, without demand to capture *all* the aspects (see below). It has been initially used as a tool to analyse students' rational behavior in proving activities according to the researchers' (and teachers') expectations (see Boero, 2006; Boero & Morselli, 2009). But its application to analyses that also use other constructs gradually developed it as a toolkit with various applications:

- To plan and analyse students' argumentative approach to the culture of theorems, in geometry and in elementary theory of numbers (e.g., Boero et al, 2010; Douek & Morselli, 2012; Morselli & Boero, 2011).

- Integrated with Toulmin and a semiotic lens, to identify different levels of awareness and control, which relate to rationality and are needed to manage arguments in advanced mathematical thinking (Arzarello & Sabena, 2011).
- To identify potential (or real) students' rationalities in elaborating arguments and face the issue of their transition to the levels and kinds of rationality aimed at by the teacher (e.g., Durand-Guerrier et al, 2012).
- To identify "ideal" rational behaviors in different mathematical fields as a means to develop teachers' awareness about the different ways of performing the same activity (e.g. proving) according to epistemic, teleological and communicative criteria (Boero et al, 2013).

Coming to the content of this RF, we may observe that the application of Habermas' construct to the analysis of mathematical activities may capture aspects that are mainly related to discursive practices, in particular those under intentional control by the subject (being she a mathematician, a student or a teacher). Other delicate issues need further elaboration: (i) the relativity of truth and the acceptability of judgments (cf. Douek and Ferrara & De Simone); (ii) the co-ordination between the creativity involved in problem solving processes and their intentional and reflective aspects (cf. Douek); (iii) the nature of cognitive processes that develop, connected with epistemic and teleological aspects of rational behavior (cf. Martignone & Sabena).

Also, Habermas considers social interaction just in relation to communicative rationality; especially, the negotiation of validity claims and the social construction of strategies are not focal points in his work. Nevertheless, his thoughts about epistemic rationality presuppose a social context for "acceptance"—at least, for a subjective presumption of "acceptance" (see above under *epistemic rationality*). We can generally recognize that Habermas does not deal with the educational problems related to rational behaviors in the classroom. Crucial aspects need further elaboration, for example: the agency of the teacher in the development of students' rationality (cf. Ferrara & De Simone); the forms of students' participation in the interaction (cf. Goizueta); teacher education to enable her to use the Habermas' construct as a tool for didactical choices (cf. Morselli et al.).

Focus on the teacher and on social interaction challenges in many ways the ideals of communication and rationality, as well as the progress of mathematics teaching and learning. The productivity of the students, in terms of effective individual learning, and that of the teacher, concerning effective creation of collective learning contexts, has to do with social aspects that intervene in broadening the reach of participation (cf. Ferrara & De Simone, and Goizueta).

The lack of interest in social interaction in Habermas is intentional and due to his effort of establishing a foundation for discursive rationality. In this Research Forum we try to put the construct of rational behavior close to that of social interaction in the planning and analysis of classroom situations with students involved in the resolution of mathematical tasks. Integrating, in a pragmatic way, an interactionist perspective with Habermas' communicative view is complex, since it requires the use of notions that were created with different purposes and within different frameworks. But we argue that it is complementary and convenient (cf. Goizueta).

The teacher should promote the development of students' rational behavior (taking care of the limits of the pertinence of her decision-making: cf. Douek). The communicative actions could be prompted by the task to be faced (e.g. request of the teacher to «explain» and «justify»; cf. Douek and Martignone & Sabena), or by the emotional engagement and expectations of the teacher (cf. Ferrara & De Simone), as well as carried on by students in interaction with peers (cf. Martignone & Sabena).

The Research Forum in its whole aims to share and discuss with participants: recent research developments and critical considerations that regard the use of Habermas for the analysis of mathematical teaching and learning (with special focus on discursive activities); and further extensions of the toolkits that have been or can be developed.

We are also interested in sharing with RF participants problems beyond what we have faced till now. We are in fact committed to identifying, revising and exploiting the potential of using Habermas in mathematics education research, even what is missing or under-theorized. In particular, once attention is drawn to dialectical relationships among participants in the activities, mutual understanding in communication is much more than just a need for guaranteeing communication. It is the result of a temporary achieved value and of a mutual acceptance. *The conditions for the creation of such value and mutual acceptance are not easy to identify and to frame whether we use Habermas' construct*. One of the tasks of the researcher, as well as of the teacher in the classroom, is to realise the value assigned to a student through the observation of how others refer to her expected productions. The ideals of communication and rationality are then linked to the capacity of altering and keeping expectations. But rationalization as an extreme and exclusive position may lead in practice to the loss of the views (and lifeworlds) of those who are not directly involved (although physically present) in the processes of reaching rational consensus. The fact that some students can be communicatively absent or unable to participate in reaching rational consensus affects the productivity of all the subjects, both teachers and students. It also fosters inequality structures. In this regard, a far goal of mathematics education research that want to take Habermas into account is *to deal with cases in which, for different reasons* (cf. Douek)*, there are students who are not discursively given opportunity to participate in the construction of shared understanding*.

[1]) Habermas (1998, p. 310) makes a distinction between behaving "rationally" (for a person who "is orientend performatively towards validity claims") and to be "rational" (for a person who "can give account for his orientation towards validity claims". Thus, criteria for the three "roots of rationality" (p. 310) establish a horizon for "rational behavior".

PRAGMATIC POTENTIAL AND CRITICAL ISSUES

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I considered the Habermas construct as sharing the modelling character of scientific constructions: it offers strategies to conceptualize classroom interaction and find ways to handle its complexity. As such, limitations may emerge at pragmatic or theoretical levels, as it is utilised.

EDUCATIONAL AIMS AND TOOLS IN EVOLUTION

I developed my reflection on the use of Habermas' construct under three assumptions:

- A vygotskian didactical perspective, conceiving teaching-learning as a dialectical construction of "scientific concepts" in relation to "everyday concepts". Among every day concepts I include spontaneous individual practices, whatever their socio-cultural roots are. Scientific conceptualization is characterized by conscious management of concepts, their properties and related practices on a general level. This perspective implies the gradual construction of class references—backing scientific conceptualization—through cycles of individual production, discussed then synthetized collectively under teacher's guidance. For scientific conceptualization argumentation is a means and an aim, as it is involved in proving and conjecturing.
- An epistemological perspective considering mathematical theories and activities as built on axioms and also on socio-cultural practices, complemented with an ethnomathematical perspective for the purpose of determining and analysing the objects and the cultural context of mathematics education.
- A conception of genuine problem solving as combining various modes of reasoning and references, not all being mathematical constructions. We can schematically identify two complementary directions: a structuring one organizing arguments and strategies, and an exploring one, when trials, metaphors, and transformational reasoning prevail. They rely on different rules of validity, but evolve dialectically.

On these bases, problem solving—in its widest meaning—is approached and enhanced through grounding activity in culturally meaningful situations for students; addressing knowledge like theorems, procedures and technical practices, and also modes of reasoning; and stimulating both creativity in exploration, and argumentation based on mathematical established references. A current didactical tool to approach these aims is based on Bartolini Bussi's (1996) construct of mathematical discussion. I interpret it as a canvas to develop students' mastery of their activity and knowledge in problem solving through two main questions: *how did you do it*? *Why is this true*? to share, criticize and develop procedures, and to identify, develop, or produce mathematical knowledge for backing and questioning certainty or data through argumentation. I consider that Toulmin's model of argumentation (1974) helps the teacher to identify the arguments' components, and to orchestrate such discussion. Toulmin's "argumentation domain" could be interpreted as the web of mathematical constructions the students have to rely upon. But, as I search a space for students' everyday conceptualization and a way to relate it to scientific conceptualization of school mathematics, I characterize "argumentation domains" by discursive practices shaped within communities sharing a cultural background (forming a cultural area of coherency).

Developing the didactical toolkit by using Habermas' construct

The idea that "*the rationality of a judgment does not imply its truth but merely its justified acceptability in a given context*" (Habermas, 1998, p. 312) fits the quest to define a wide argumentation domain to back a diversified epistemic component. Moreover, this construct allows discussing teleological reasons and communicational choices. Hence the idea of developing the mathematical discussion along the three components of rationality. On a more general level, this construct allows to identify fine grain components of potential argumentation lines enriching problem solving, and to identify different levels of rationality within the curriculum development from one school level to another, and different kinds of rationality according to different mathematical domains (cf. Morselli et al.). Thus it enriches the potential of mathematical discussion and allows improving its planning and management. My interpretation of the Habermas construct affects the articulation of the above mentioned educational choices. To create a classroom context suitable to promote the vygotskian dialectics and develop scientific conceptualization and argumentation, supposes to: 1) introduce students to cultural interest for finding reasons that have a theoretical relevance and/or can be shared as valid references; 2) make them aware that reasons can be various, not all based on the class references, and understand the relations between those and the specific references related to everyday concepts; 3) develop attention and concern for interaction and ability to adequately express one's views in a given socio-cultural context; 4) develop consciousness of one owns positioning and a critical attention to it; 5) establish mathematical references collectively—theoretical knowledge and practices—under teacher's guidance, built upon various sources, including students' contributions and cultural experience.

To reach such aims, the teacher needs to stimulate and offer a model of rational behaviour and discourse (according to the school level and the specific subject) and of acceptance of a variety of justifications related to a variety of backings—as long as they are made explicit. The corresponding development of the didactical toolkit can be presented as a canvas of ***rational questioning*** to organize the mathematical discussion according to the three components of rationality as a way to introduce the students, and lead them, to behaviours shaped by rationality requirements: *why do you think that it is true? Why do you need to do that for...? Did you make yourself understandable to...? Did you (the others) understand why she…, how he…,* etc. And in order to allow relativization of justifications to an argumentation domain (or level):

is this reason different from…? From his point of view could you say the same thing? In this way, students' voices are received and deepened, bringing as much as possible their roots to consciousness and making some links emerge, in particular with class references. The teacher can support students' "rationalization" of discourse (i.e. afterward fitting rationality requirements) and the relativization of reasons according to references collectively put to light, dialectically forming argumentation domains.

CRITICAL CONSIDERATIONS

The requirement of intentionality and awareness from the learners' part

To sustain students' scientific conceptualization means to draw them to be able to refer to the classroom (or some outer acknowledged) constructions as theoretical references, mobilize them consciously, and understand their potential generality. This attitude in solving a problem is difficult to handle in the beginning of the process, while exploring the situation. Nor is it easy to establish a strategy while trying to find one's way. This is the case, whether the students try to rely upon classroom constructions or on specific local references. Genuine problem solving generally needs stimulation from others. Rational questioning should favour student's maturation (from the ability to act and develop a discourse about acting towards the ability to organise strategies and express them a priori, when the situation is mastered enough) by supporting—in between—going back and forth from "action" to its rationalization, accounting for validity of statements and strategies, and produce autonomously a conclusive rational discourse. Such a sequence should include explicit relating, contrasting and combining organization aspects of activity and exploration (which may involve everyday conceptualization) in order to develop problem solving abilities, on a long term perspective.

The "race to 20" example presented by Martignone and Sabena shows a didactical situation where the gradual transition from exploration to a rational attitude to organize explanation takes place (the teacher relies upon a didactical contract inducing students' efforts of explanation). Explanations cannot occur at the first trials. Students move gradually from playing the game, to describing and making claims, then to justifying them. When the teacher encourages them to organize and generalize the justification, they attain both a justification of why the solution works (based on arithmetic knowledge), and an organized description of the activity bearing a general character: they produce a rhythmic exposition of the game moves, with voice and gesture, that point to the various "variables" affecting the moves, and to hierarchy on their treatment. They gain ground along the three components of rational behaviour once they had explored and then gradually move to accounting for strategies and related reasons of validity.

Can we always recognize the rationality of another? The necessity of doubt

Excavating reasons and elaborating convenient communication modes may induce a real logical muddle. The reasons a person produces—after or before action, as well as communication efforts, depend on her perception of the context and the related argumentation domain(s), and on her perception of the listeners' expectations—that may not coincide with hers. As I understand Habermas' construct, those speaker's perceptions are essential elements for her intentional choices (intentionality is a criterion of rationality). Can the listener/observer be sure the speaker searched for reasons and convenient backing, and correctly perceived the listeners' expectations? Are they able to grasp them? Who can legitimately control validity criteria?

In an educational context, can we be sure we are able to welcome students' efforts of rationalization? Don't we risk inducing them to adopt non significant references and to impose links that hinders coherency between cultural roots—that they may not master consciously—and school's constructions? The issue is difficult to deal with. Teacher and student's positions are not symmetrical, and agreement can result from an authoritarian process instead of cooperative rationalization. But education is an insertion into a cultural community! The difficulty is striking when educators and students do not share a common culture. Moreover, there can be a gap between possibilities of activity and possibilities to develop a related discourse, especially in exploration: the student may lack discursive ability; or not grasp all the reasons behind one's behaviour because conceptualization is not yet sufficiently developed (e.g. elements affecting reasoning are not all conscious); or have deep difficulty to express oneself through a logically organised discourse (e.g. transformational reasoning is not easy to justify, and even to describe completely).

On the didactical level, all these difficulties do not imply to renounce to rational questioning within class discussions, but to seek a way to give space to doubt and to suspend conclusions, as a collective agreement that needs to be shared sometimes.

Recognizing students' rationality is almost always a challenge. In a teaching experiment Habermas' construct was used to analyse secondary school students trying to elaborate a proof (see Morselli & Boero, 2011; Douek & Morselli, 2012). A game was used to introduce algebra as a tool for proving: they were asked to choose a number then transform it through prescribed simple calculations. The transformation eliminated the chosen number from the final result. They had to explain why the result is constant whatever number is chosen. Two productions were exploited. Ric's explanation was close to an algebraic expression. It was analysed as based on adequate epistemic reasons, produced according to efficient teleological choices and well communicated (though many schoolfellows did not grasp it). Tor's reasons were better understood by his schoolfellows; his procedure to prove was well organized, but his calculations were incorrectly formulated (according to standard syntactic requirements). The first analysis pointed to a lack of epistemic rationality. But deeper reflection showed that his organization of calculations resembled computer programs (systematically naming X the result of each calculation step). Somehow, the process of analysis moved from an implicit and unconscious relativization to the targeted algebra field towards a conscious relativizing of the analysis to a new specific argument domain. Within the wider epistemic component including computer science, Tor's expression was almost correct, and revealed "good" epistemic basis. This critical reflection permitted to legitimate a potential epistemic rationality the student could not uncover, and to express its potential "scientific coherence". But it was not possible to share with the class this structuration of his justification—as coherently related to a system. In general, to identify students' potential rationality, welcome it, and relate it with classroom construction is a difficult challenge. Here, the teacher brought most students to understand both argumentations, but put to light only the pertinence and validity of the "algebraic" production according to the requirements of the mathematical community, on the epistemic and teleological sides.

Problem solving creativity and communicative rationality

Problem solving intertwines creative exploration and rationalization. Exploration needs to be freed from stereotyped modes of reasoning, and to evolve through doubt, using uncertain metaphors, approximate semiotic representation, transformational reasoning, interpretation and links. This relates to communicating with oneself. The rational questioning, be it the student's reflective activity or be it resulting from interaction should be a method in organization phases, on the structuration side. On the exploration side, it should allow to bring to consciousness creative behaviour; and to elaborate a critical view on innovative ideas, and on the necessity, difficulties and benefits of combining structuration and exploration. But it may hinder aspects of exploration and disturb interpretation and communicating with oneself, in transformational reasoning or in producing metaphors for example. It may happen that the need for validating statements results in the uncritical use of established knowledge; and that aiming at correct solutions results in the search for established methods. These well known phenomena depend on the didactical contract. They may be related to the lack of acknowledged space for doubt, to teacher's premature or misleading request, and/or request at inopportune phases. Thus, creative exploratory phases need a flexible exploitation of rational questioning, allowing didactical treatments ranging from releasing from justification and accepting doubt, to efforts to remove doubt accompanied by teacher's care for genuine students' accounting—at the convenient moment—for validity of statements and strategies.

THE EMERGENCE OF VALIDITY CONDITIONS IN THE SECONDARY MATHEMATICS CLASSROOM: LINKING SOCIAL AND EPISTEMIC PERSPECTIVES

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INTRODUCTION

It is widely accepted that argumentative competencies should be developed within the mathematical activity in the classroom, both as a product of this activity and as a means to support it. We share Boero's (2011) idea that a 'culture of argumentation' is to be developed in the classroom and that it should include practices on the production of conjectures, meta-mathematical knowledge about the acceptability of references advanced for the validation (acceptance/rejection) of claims and knowledge about the role of counter-examples and generality. It should include elements for evaluation of mathematical productions and general ideas about the use of all this knowledge within argumentative practices, along with the needed awareness to allow deliberate and autonomous control of the process.

In our study we address some specific aspects of the broad challenge of fostering such culture in the classroom. Our main interest is to investigate the epistemological basis of argumentative practices in the mathematics classroom and, particularly, how is validity interactively negotiated and constructed, as a rational enterprise, in a rich problem-solving mathematical activity. In the following, we will show how Habermas' construct of Rational Behavior is used for that purpose within our study and how we complement it with other theoretical constructs in order to better suit our needs, accounting for the social and epistemic complexity and specificity of the mathematics classroom. This theoretical integration frames our understanding of classroom argumentative practices and gives us a tool for investigating epistemic features of these practices in order to analyse data coming from students' interaction.

THEORETICAL FRAMEWORK

Following Steinbring (2005), we do not understand mathematical knowledge as a pre-given, finished product but, instead, as the situated outcome of the epistemological conditions of its dynamic, interactive development. We assume that a "specific social epistemology of mathematical knowledge is constituted in classroom interaction and this assumption influences the possibilities and the manner of how to analyse and interpret mathematical communication" (p. 35). Within this socially constituted mathematics classroom epistemology a criterion of mathematical validity is interactively negotiated between the participants. Mathematical activity and mathematics classroom epistemology are reciprocally dependant: the later shapes a frame in which the former takes place and the former develops the later to conform to the emergence of new legitimated mathematical (and meta-mathematical) discourses. A central consequence of these assumptions is the basic necessity for interpretative research to reconstruct the situated conditions in which (and from which) mathematical knowledge is interactively developed. Although 'conditions' might be considered in a broader sense, we are particularly interested in the epistemological assumptions at stake, the references (mathematical and not) that might be considered as relevant and the social environment in which the process is embedded.

According to Habermas' tridimensional description of rational behavior (Habermas, 1998), a person acts rationally when she is able to explain in a reflective attitude (and thus is aware of) how is her action guided by claims to validity, accounting for what she believes, does and says in accordance to the intersubjectively shared culture. If we think of the mathematics classroom as a social environment in which knowledge is interactively constructed according to evolving epistemic specificities and in which a particular culture of argumentation is to be intentionally constructed, Habermas dissection of rational behavior constitutes a very appealing descriptive tool. It allows focusing on specific features and issues at stake, and to plan and exercise adequate control to foster culturally accepted practices, promote noetic transparency and allow students to gain awareness about the intended culture.

Adapting Habermas construct to the classroom

In the situated context of the mathematics classroom, the general relation between classroom epistemology, mathematical activity and social environment must be considered under the light of a specific, content-related didactical contract (Brousseau, 1997). The didactical contract corresponds to the reciprocal expectations and obligations perceived within the didactical situation by the teacher and the students with respect to the knowledge in question. Mathematical acceptability of students' explanations is linked to these expectations and to the mathematical contents at stake (or perceived as being at stake). Thus, when faced with a problem, students might bring up mathematical knowledge and references they consider relevant for the proposed didactical situation in order to cope with the task; a common clause of the didactical contract may indicate them to do so. Nevertheless, not all the emerging references are linkable to well established and intersubjectively shared mathematical knowledge. We might also need to consider other references (statements, visual and experimental evidence, physical constraints, etc.) that are not part of institutionalized corpora, such as scholar mathematics, and are used de facto as taken-as-shared, unquestionable knowledge (Douek, 2007). A contextual corpus of references is necessary to support everyday argumentation but also mathematical argumentation at any level; it might be tacitly and operatively used by the students to make sense of the task, semantically ground their mathematical activity and back their arguments. Accounting for the reference corpus at stake might be particularly relevant when considering problem-solving settings in which empirical references are to be considered as part of the proposed milieu.

By complementing Habermas' notion of rational behavior with this situated view of the students' activity and the classroom social and epistemological environment, we try to better understand the argumentative practices of the classroom and to inductively identify underlying epistemic constraints that might be behind observed practices students enact to validate their arguments. Because these practices are multifarious and often implicit in the interaction, specific analytic tools are needed to observe the conversational level with the needed focus and detail.

Adapting Habermas construct to analyze the conversational level of interaction

Toulmin (1958) structurally describes an argument as consisting of some data, a warrant and its backing and a modalized claim. The backing supports the warrant, which allows inferring the claim from the advanced data. Conditions under which the warrant does not support the inference (rebuttal conditions) might be also considered.

Toulmin's model allows one to focus on arguments as the basic analysis unit and to structurally dissect them, while Habermas' construct allows one to focus on the main motive of the ongoing discursive activity and apprehend those structural parts on their epistemic, teleological and communicative dimensions. We use Toulmin model strictly in order to structurally describe argumentation and account for validity evaluation in accordance to our theoretical perspective. According to Habermas (1998), accepting a validity claim is tantamount to accepting that its legitimacy can be adequately justified, that is, that conditions for validity may be fulfilled. Under our theoretical perspective, validity conditions are explicit or tacit constraints that allow students to control the coherence of the mathematical activity according to the socially constituted classroom epistemology, didactical contract, reference corpus and shared goals. Our main interest is to reconstruct the emergence of validity conditions and the process of fulfilment of this conditions, be it successful or not, as they are brought up by students as means for validation.

Because this analysis mainly occurs at the conversational level, we incorporate yet another analytic tool in order to focus and apprehend particular features of the conversation. According to Sfard and Kieran (2001) an exchange is considered effective if "all the parties involved view their expectations as fulfilled by the interlocutors" (p. 51). They propose to assess effectiveness through the analysis of the 'discursive focus': a tripartite construct consisting of a 'pronounced focus', what is actually said (and is thus public), an 'attended focus', what attention is directed to (including the attending procedure) and an 'intended focus', "a cluster of experiences evoked by the other focal components plus all the statements a person would be able make on the entity in question" (p. 53). Through this 'focal analysis', referred to our theoretical perspective, we bring to the fore particular aspects of the conversation that might be indicators and descriptors of the mainly tacit emergence and fulfilment of validity conditions. Through the inductive-interpretative analysis of discursive foci we identify the illocutionary intention of establishing and fulfilling validity conditions in order to perform the illocutionary speech act of validating claims.

THE DESIGN EXPERIMENT

We conducted a design experiment were thirty 14/15-year-old students and their teacher worked in two lessons in a regular classroom in Barcelona, Catalonia-Spain. It was a problem-solving setting, with time for small group work and whole-class discussion. The following problem was suggested by the researchers and intended as an introductory task to probability theory:

> Two players are flipping a coin in such a way that the first one wins a point with every head and the other wins a point with every tail. Each is betting €3 and they agree that the first to reach 8 points gets the €6. Unexpectedly, they are asked to interrupt the game when one of them has 7 points and the other 5. How should they split the bet? Justify your answer.

The novelty of the task was expected to lead students to develop models and negotiate meanings, while producing arguments to validate them avoiding mechanical approaches based on well-established heuristics. The teacher was asked to avoid hint-based guidance and favor reflection by proposing different numerical cases. Crucial cases were planned to problematize expected wrong proportional answers. For data collection, two small groups were videotaped and written protocols were collected. Some groups were collectively interviewed a week after the task.

OVERVIEW OF THE PRESENTATION WITHIN THE RESEARCH FORUM

We will present the case of a group of four students working on this problem. Drawing on a proportional model (corresponding to the points won), they come up with the solution: €3.5 for the winning player and €2.5 for the other. This model emerges as the result of the fulfilment of certain validity conditions. The teacher asks them to “check different situations to see if that reasoning holds” and proposes the case '2 points to 0'. This case was expected to problematize their model by producing a counter-intuitive result, driving the students to seek for a better fitting new model.

During the interview students were asked to watch on video this episode and then were asked about the relevance of testing the model in different numerical situations.

1 R:	So, the teacher comes, you explain to her what's going on and then she asks you to test it in other situations. Why do you think she asks for this?
2 Zoe:	To check it. If it happens the same.
3 Josy:	To check it.
4 Vasi:	If it always holds up. I mean, if it is not only in this case.
5 R:	And why would it be important that it 'holds up' in other cases?
6 Anna:	Because that way you verify that your method (…) is correct.
7 R:	And what does it mean for a method to be correct?
8 Anna:	Well, in this case, that the distribution is fair for both of them, and that it works not only in this case but also in others.

By directing the attention to the teacher's request, the intended focus of the interviewer in 1 is on the relation between particular case exploration, generality and explanatory power of the model when considering empirical situations. The students do not point, as was expected, to the necessity of the model to accord to shared empirical references about the game. Instead, our analysis shows that the deictic used by Zoe in 2, which corresponds to her intended focus, refers to the validity conditions that originated the model and their fulfilment. These conditions are intrinsic to the model and thus their fulfilment is guaranteed. In 6, Anna considers the fulfilment of the developed validity conditions as sufficient to positively assess the model as “correct” (although the epistemic status of the claim is not clear) and, in 8, makes 'correction' equivalent to 'fairness' in this case. By operationalizing the notion of fairness in terms of the developed validity conditions the requested testing of the model becomes a self-fulfilling process that inductively reinforces the model. This explains why these students support a counter-intuitive result when working with the example '2 points to 0', instead of problematizing the model.

In our presentation we will develop this example and use our analytical approach to show how the epistemic and social conditions of emergence of the model end up constituting an obstacle that prevents students from challenging it.

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ANALYSIS OF ARGUMENTATION PROCESSES IN STRATEGIC INTERACTION PROBLEMS

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In this study we use Habermas construct to analyse argumentation processes related to strategic interaction problems. These problems provide suitable environments to develop and analyse students' planning and control processes, of a paramount importance in mathematical problem-solving. Some theoretical tools to study planning and control processes will be integrated with Habermas construct through the analysis of excerpts from a classroom discussion in grade 4. This integrated analysis will highlight specific features of the argumentative discourses, brought to the fore in strategic games.

STRATEGIC INTERACTION PROBLEMS

In strategic interaction problems, two or more decision makers can control one or more variables that affect the problem results. The decisions of each player influence the final result of the game. Game Theory offers different mathematical models for the winning strategies, based on specific assumptions about how ideal, hypercalculating, emotionless players would behave (Von Neumann & Morgenstern, 1947). However, analysing strategic interaction games as problem-solving activities, Simon (1955) argues that the limited capabilities of the human mind (memory system and the development of calculus, attention span, etc.) combined with the complexity of the external context, make often impossible the elaboration of the strategic choices predicted by Game Theory. As concerns limits on iterated thinking, the data collected by Camerer (2003) show that, during the resolution of strategic interaction problems faced for the first time, only few subjects are able to develop many thinking ahead steps (*limited strategic thinking*).

PLANNING AND CONTROL IN MATHEMATICAL PROBLEM SOLVING

Our research is based on the assumption that strategic interaction problems constitute suitable environments to develop and analyse important features of genuine problem-solving, such as *planning* and *control processes*. Planning processes are related to the possible actions to be performed across time: for this reason, the studies about mind times can be useful to interpret the specific related cognitive processes. Considering problem-solving activities, Guala and Boero (1999) identify some examples of mind times (i.e. time of past experience, contemporaneity times, exploration time, synchronous connection time) involved in the imagination of possible actions over time. In particular, they analyse the *exploration time* as the projections that can be developed from the present onward (e.g. “which strategies can I develop to find the solution?”, “How can I manipulate the data to solve the problem?”) or from the future back to the past (e.g. “I think up a solution and explore it in order to find the operations to be performed, depending on available resources”). Planning processes can be analysed deeper by taking into account the cognitive studies about the human ability to remember (Tulving, 2002) and imagine facts and situations in the course of time (Martignone, 2007). “Remembering” and “projecting” need the ability to conceive *the self* in the past and future, which goes beyond simple “knowing” about past events and future facts. In particular, considering the imagination of possible future events, we can distinguish between the knowledge that we possess about an event (*semantic future thinking*), versus thought that involves projecting the self into the future (*episodic future thinking*) to “experience” an event (Atance & O’Neill, 2001). Knowledge supports and structures imagination processes through frames or scripts that influence the expectations on stereotyped situations.

Note that in episodic future thinking the imagination is not given free reign, but rather, the projection is constrained. For instance, envisaging my forthcoming vacation might require me to consider such factors as how much spending money I will have, how much work I will have completed before I go, and so on. (*ibid*., p.533)

Besides planning processes, also control processes play a fundamental role in problem-solving activities. As introduced by Schoenfeld (1985), control deals with “global decisions regarding the selection and implementation of resources and strategies” (p. 15). It entails actions such as: planning, monitoring, assessment, decision-making, and conscious meta-cognitive acts. In the context of argumentation and proof activities, Arzarello and Sabena (2011) show how students’ processes are managed and guided according to intertwined modalities of control, namely semiotic and theoretic control. *Semiotic control* relates to knowledge and decisions concerning mainly the selection and implementation of semiotic resources. For instance, semiotic control is necessary to choose a suitable semiotic representation for a problem (e.g. an algebraic formula vs a Cartesian graph). *Theoretic control* requires the explicit reference to the theoretical aspects of the mathematical activity: it intervenes when a subject use consciously a certain property or theorem for supporting an argument.

Our study integrates the identification of planning and control processes in strategic actions development, with the study of the communicative actions as rational discourses according to Habermas construct (cf. Boero & Planas).

METHODOLOGY

On the base of the theoretical discussion above, teaching-experiments are planned and analysed with the collaboration of classroom teachers. Activities develop around classical strategy games, such as NIM, Chomp, Prisoner Dilemma, etc., and alternate game phases with reflection phases (collective discussions, written reports). Video-recordings of the discussions and students' written reports are collected and analysed on a qualitative and interpretative base.

In the following we consider a case study in grade 4, based on a strategic interaction game called "Race to 20", which was used by Brousseau to illustrate the Theory of Didactical Situation (Brousseau, 1997). The rules of the game are the following. There are two players: they know the possible alternative choices and the relative outcomes, they do not cooperate and do not know in advance the adversary strategies. The first player must say a number between 1 and 2. The second player must add 1 or 2 to the previous number, and says the result. Now the first player adds 1 or 2, and so on… The player who says 20 wins the game.

ANALYSIS OF THE COLLECTIVE DISCUSSION

We analyse some excerpts of the collective discussion organized by the teacher after the children have played the game several times, at first individually, and then in teams. In the discussion, students are asked i) to describe possible winning strategies and ii) to justify them. Numbers 14 and 17 are soon identified as "winning numbers". Justifications are based on the possible moves of the two players. We report Elena's contribution:

Elena: it's necessary to arrive before at 14 and then at 17. Because if you do from 14 you do plus 1 and arrive at 15 and then you do plus 2 and arrive at 17, which then…you do plus 1 and arrive at 18 and the other does plus 2 and arrives at 20. Rather, if you do plus 2 from 14, you arrive at 16 and the other does 1 and arrives at 17, the other if he does plus 2 arrives at 19, you do plus 1 and arrive at 20. Hence anyway from 14 to 17 you arrive anyway at 20.

Elena carries out her argumentation by describing the possible moves of the players (*episodic future thinking*) who start from two particular positions (14 and 17). The *teleological aspect* is clear: she wants to describe the winning choices. Because the steps of thinking are limited (*limited strategic thinking*), she manages to plan ahead only close to the winning end, i.e. number 20. When the teacher asks the students if there are other numbers like 14 and 17, different scenarios are explored by students. The number line helps them to *control* the winning positions and strategies, relying on the semiotic representation at the blackboard (Fig. 1).

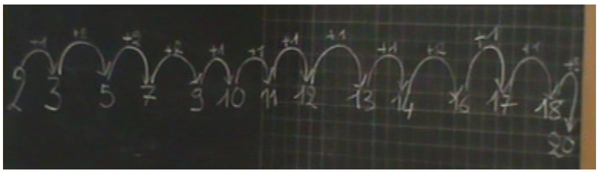

Figure 1. The written representation used to play the game.

The students' attention is focused both on backwards movements, in search of previous winning position, and on the forward movements, to check the efficacy of the elaborated strategies.

Diego: 11 maybe is an important number, because maybe my team adds 2 and it is 13, the other team adds 1 and arrives at 14, I add 1, 15, they add 2 and it is 17

Elisa: but if they are stupid they make plus 1 and arrive at 18, we make plus 2 and arrive at 20; but they are not so stupid to do plus 1, eh!

Also Diego and Elisa rely on the *episodic future thinking* to imagine the possible moves of the players and justify their hypothesis about number 11. The steps of thinking ahead are always "close" to the new winning numbers (*limited strategic thinking*). In the rational discourse of Elisa, the *teleological component* is linked to the goal of the game—getting to number 20—and it is guided by her knowledge (stressed in the discourse) that the other players have the same information and capacity. After that many students have expressed similar arguments, Giulio proposes a general rule, which can drive all strategies:

Giulio: I think that for the winning numbers you always remove 3: from 20 you remove 3 and you arrive at 17; from 17 you remove 3 and you arrive at 14, I think that another winning number could be 11, could be…8, could be…5, could be…2

Teacher: Explain well this idea

Giulio: Because…that is I don't know, if I arrive at 2…I don't know, I begin, I make 1, no I make 2, he arrives and makes 1 (gesture in Fig. 2a), I put 2 and I arrived at 5 (Fig. 2b), which I think is a winning number… yes, arrived at 5…it is a winning number, I think. Then…he adds 2, say (Fig. 2c), I add 1 and I arrived at 8, which is another winning number. He adds 1, I add 2 and I arrive at…12, which is another winning number. He adds 2, I add 1, and I arrived at 14, which is another winning number, he adds 1 I add 2, we arrive at 17 which is a winning number, he adds 1 or 2, I add 1 o 2 and I win

Figure 2a,b,c. Giulio's gestures in his argument.

In his first sentence, Giulio expresses the rule in a general way, as an a-temporalized relationship between numbers ("you always remove 3" from 20). Giulio's argument is based on the backward induction (similar to what described in Game Theory): he starts from the winning result and moving backward he identifies the best strategy to win the game. In the argumentative discourse on the winning strategy, *the teleological and epistemic components* of rationality are on the foreground. The epistemic plane relies on the relationships between numbers, and in particular on the (both *semiotic* and *theoretical) control* over the number line model, recalled in the written schema introduced by the teacher to play the game (Fig. 1). This representation has now become a thinking tool for Giulio. Asked to better explain his ideas (*communicative component*), the boy imagines a match, and describes the moves in a temporalized way (*episodic future thinking*). The subtraction turns into an onward movement starting from the very first move (number 2). This movement is produced by means of a rhythmical repetition of the same linguistic structure: "he adds…I add…and I arrive at…, which is a winning number". Linguistic repetition is co-timed with gesture repetition during the entire argument: gestures are synchronous with the added and obtained numbers in the imagined game. Gestures and words together constitute a schema through which the generality of the argument is conveyed. Gesture in Fig. 2a (open hand as holding something) is co-timed with the words "makes 1": while saying "1", Giulio is indeed meaning "let's say 1" or "any move of the player", something similar to what Balacheff in proving processes called "generic example" (Balacheff, 1987). This interpretation is confirmed, besides by the voice intonation, by the gesture-speech combination of Fig. 2c: a similar gesture is performed within a similar speech schema, but now the generic nature of the example is made explicit by the word "say".

CONCLUDING REMARKS

In this paper we analysed some excerpts of a discussion about the winning strategies in a particular strategic interaction problem: the Race to 20. It provided a suitable context to study the students' argumentations, intended as rational discourses in the Habermas construct. The analysis integrated the cognitive studies about mind times, limited strategic thinking and semiotic aspects of control processes with the study of communicative actions. As a result of the use of these different interpretative tools, an important distinction in the teleological component of the argumentations emerged. In fact, we can identify two teleological planes: a *pragmatic* plane, related to the goal of the game ("Which strategy can I choose or develop in order to win the game?"), and a *theoretical* plane, related to justifying the chosen strategy ("How can I justify that my strategy is the best one?"). Furthermore, as we could see in the reported excerpts, the two planes are deeply intertwined: theoretical considerations can fruitful ground on pragmatic ones, and—more important—can also be justified on a pragmatic base (see Giulio's argument and the generic example therein). This feature is inherent to the specific didactic engineering based on strategic interaction problems: besides the game phases, in fact, it is the *request of sharing and justifying* *their strategies* what makes these activities a suitable context to develop communicative actions, as rational discourses, from early school grades. As a matter of fact, on the *theoretical* plane, the *teleological* dimension strongly intertwines with the *communicative* one, when students are asked by the teacher to explain and justify their strategies to their mates (for the analysis of possible teacher's intentions see Morselli *et al.*, in this RF).

USING HABERMAS IN THE STUDY OF MATHEMATICS TEACHING: THE NEED FOR A WIDER PERSPECTIVE

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THE TEACHER AS A RATIONAL BEING

In the last decade, Habermas' notion of rationality has received lots of attention from mathematics education research whose main focus is on the learners' argumentative activities. In one chapter of the book *On the pragmatics of communication*, Habermas (1998) offers an analysis of the complementary relationship between the discursive activity of a rational being and the reflection on it. Drawing on this, we argue that we need to grapple with the mathematics teacher *as* a rational being, who participates in discursive activities and whose orientation towards validity claims has to do with her decision-making. Our aim is to examine the *'rational being'* of the teacher in the classroom rather than the thinking processes of individual learners.

Following Habermas in recognizing the complementarity above, we propose that, for talking about the teacher's rationality in her decision-making, we need to consider reflection on personal activity, what involves values and beliefs of the acting subject. This entails a true complexity for our study, since the beliefs and the background of the teacher contribute to her choices. The frame is even more complex whether we accept that beliefs are *not* agent-neutral, which means that the affective sphere of the teacher also affects her beliefs.

So, *what happens* in the mathematics classroom is only a part of the story. We need a wider perspective, which leaves room for *the feeling of what happens* that the teacher brings to activity. Briefly speaking, if we remain clung just to what happens we might loose the reasons for which that specific 'what' happens in the way it does. There is here an implicit assumption: what happens in the classroom is entangled with feelings of what happens that can be ascribed to individual teachers. Our perspective to study rationality in mathematics teaching expands to include also the affective domain, especially the 'emotion side'.

RATIONALITY AND EMOTION

The recent research on affect in mathematics education has increasingly recognised that affect and cognition are strictly related and unlikely separable (e.g. Zan et al., 2006; Hannula, 2012). Other than emphasizing the richness of the field through a variety of perspectives, many studies have marked the delicate role of *emotion* in the landscape of affect. Zan et al. (2006) have pointed out "how repeated experience of emotion may be seen as the basis for more 'stable' attitudes and beliefs" (p. 116). Hannula (2012) affirms: "people can have very stable patterns for emotional arousal across similar situations, which is the foundation of the whole concept of attitude" (p. 141). Mainly in the same years, some criticism of Habermas' work has highlighted the demoted role given to affect-related aspects:

> "However, as soon as not just purely theoretical questions but practical ones are concerned—questions that bring values, attitudes and emotions into play—agreement will not be reached exclusively through arguments, as Habermas demands of all agreement reached communicatively—but rather […] through all sorts of non-argumentative means of influence, such as the way arguments are presented, affection or dislike for the one presenting the argument, unconscious group dynamics, etc. There is not, in point of fact, any agreement in practical questions where such factors do not play a role." (Steinhoff, 2009, p. 205)

By their very nature, emotions are closely connected with both social systems and the biological human body. Research in neuroscience has supported evidence of the interplay among cognition, metacognition and affect. In particular, the studies of Damasio re-evaluated the role of emotion and feelings in decision-making: "certain levels of emotion processing probably point us to the sector of the decision-making space where our reason can operate most efficiently." (Damasio, 1999, p. 42). More recently, Immordino-Yang and Damasio (2007) have considered the relevance of affective and social neuroscience to education, proposing to consider emotion as a "basic form of decision-making, a repertoire of know-how and actions that allows people to respond appropriately in different situations" (p. 7).

As the perspective is widened to count emotion with respect to the teacher's beliefs and decision-making, the rationale of our study becomes clear. But we have sort of face questions like: How can we talk about the discursive activity of the mathematics teacher taking into account her emotional engagement in it? How can we talk about the *entanglement* of rationality and emotion in mathematics teaching? Looking at mathematics education research again, we have found a possible answer drawing on Brown and Reid (2006)'s study, which offers the notion of *emotional orientation* as a theoretical construct to analyse teachers' decision-making.

EMOTIONAL ORIENTATION(S)

Brown and Reid's study is relevant for our research for its interests in that particular emotional aspect of human behaviour, which the authors feel as neglected and see as "related to the decision-making that happens before conscious awareness of the decision to be made occurs." (p. 179). Following Maturana (1988), Brown and Reid refer the idea of emotional orientation to the criteria for acceptance of an explanation by members of a community and emotions to the foundation of such criteria. The criteria for accepting an explanation (*x*s) *cannot* be the same as the criteria for accepting the criteria ('meta-criteria' *y*s). We can draw on this distinction to interpret emotions as being at the subtlest degree, that is, as *moving* those *y*s for accepting the *x*s, figuring out emotional orientation as set of meta-criteria. We stop here in order to avoid an infinite regress. So, the teacher's rationality will allow us to talk about what happens—in terms of *x*s, while her emotional orientation will inform us of the feeling of what happens—in terms of *y*s. The two aspects are purely *intertwined* and joined in a unique frame that intends to speak directly to mathematics teaching in contextual situations. But we need to identify the emotional orientation of the teacher. Recalling Damasio, Brown and Reid introduce somatic markers as those structures that inform our action and decision-making, pushing us to decide something since "it feels right"—in terms of its acceptance in a community. In decision-making, they say, "many possibilities are rejected because they are associated with negative somatic markers" (p. 180), while *positive* somatic markers entail possible behaviours that reveal the teacher's decisions in the activity. They see emotional orientation as set of somatic markers, to which emotions related to *being right* are attached.

Following Brown and Reid in seeing 'the being right' as crucial, we characterize the emotional orientation as follows. We focus on the teacher's beliefs concerning the context, the content, the subject matter and her experiential background—beliefs that she declares in an a-priori interview. We identify her *expectations* concerning the activity—expectations that are attached to the beliefs and that we recover from videos of her actual activity in the classroom. The word 'expectation' is used for its positive meaning of wait and anticipation, which we can refer back to emotions of being right. Briefly speaking, the set of expectations shapes the teacher's emotional orientation, which entails belief-related actions that reveal the rationality of her decision-making. In the next section, we illustrate the example of a teacher.

THE EMOTIONAL RATIONAL BEING OF CARLOTTA

Methodology. For space constraints, we only sketch the context and methodology of the study. The study is part of a wider research, whose focus is on the rationality of the teaching of linear equations at secondary school, and involves 3 teachers and their grade 9 classrooms, in Northern Italy. Each teacher was first interviewed and asked about her beliefs on linear equations and algebra in general. The interviews were twenty minutes long and were videotaped with the camera facing the interviewer and the subject. The teachers' activity in the classroom was also videotaped. The videos were transcribed for data analysis.

Classroom culture and knowledge stability. During the interview, Carlotta said:

> "For me (*sighing*), the greatest problem I'm trying to solve—I realised, in these last years it's becoming tragic—is the problem of the stability of knowledge; I feel that, in many

classrooms, (*speeding up*) apart from the good ones, students don't remember what we did and for me this is serious. For example, in grade 10, I'd like to refer to something that I did in grade 9, on which I've even insisted, without having to repeat it entirely… the big problem to solve, in which I persist a lot, is to be able to find a way to construct a core (*miming a base*), a base of knowledge (*miming a list*), of abilities that stay. For me, aside from time economy—'cause, maybe, it's annoying having always to recall—it's really a matter that has to do with cognitive science, I don't know, I wouldn't know how to face it well, but it's becoming a generalised problem, then… we should look for, the problem is looking for meaningful activities that allow for… fixing things."

Carlotta feels that knowledge stability is a "serious problem" in her teaching and she sees classroom culture as a possible solution for having a "core of knowledge". From this belief, she *expects to construct new knowledge from what has been already done in the classroom*, as it is shown by many classroom moments. An example is given when Carlotta decides to present the "properties of linear equations" starting from the laws for equalities that were introduced at the beginning of the year (and that regard the substitution properties: adding/subtracting a number to—multiplying/dividing by it—both sides of an equality does not change the equality):

Carlotta: How do the laws for equalities translate into properties for equations? (*forward-facing, with a hand on the desk, raising her eyebrows*) What can you say? … That, if you have an equation, right? What do you do?

S3: If we multiply or divide both sides of an equation by the same value, we will get an equivalent equation

S8: Even subtracting we get…

Carlotta: Let's say: For the first law, given an equation, if we add the same number to both sides or we subtract […] it means to sum the opposite, right? We can speak of sum. Then, if we sum both sides of the equation (*miming them with both hands*) we get (*nodding, waiting for the students to speak*)

S6: An equivalent equation

Carlotta: An equation equivalent (*nodding*) to the given one. Instead, for the second law… (*nodding, biting her close lips, gesturing a fist in the air*; Fig. 1)

S3: If we multiply or divide (*Carlotta nods, keeps her lips close and the fist in the air*; Fig. 1)

S5: By a number different from zero

S7: Both sides

S3: We get an equation that is equivalent to the given one

Figure 1. Carlotta's expression and gesture

A fabric of emotion and rationality. In this context, we talk about the entanglement of emotion and rationality by looking at how the expectation about classroom culture narrows, marking Carlotta's *positive* emotion in constructing the properties of linear equations from the known laws for equalities: "for the first law, given an equation". Emotions are revealed by her *somatic* engagement—gesture, gaze, head movement, facial expression, tone of voice—in interaction with the classroom. Carlotta creates an *intersubjectively shared* space where to encounter the students and make them to disclose the same connection through what they know ("What can you say?"), which represents their horizon of reference. Her *epistemic rationality* refers to the nature of the laws for equalities—like, in the case of the first law, the possibility of speaking always of sum, instead of distinguishing between adding and subtracting—and brings at play the relationship with the properties of linear equations. This generates actions to accomplish the goal: having students to discover the connection and apply it for gaining an "equation equivalent to the given one" (*teleological rationality*). Carlotta uses many ways to communicate: gestures, gazes, facial expression and tone of voice changes, pauses and demands. The repeated use of the pronoun "we" is also part of her *communicative rationality*, and reveals the attempt to actively engage students, as well as her participation to learning construction—often expressed with nodding.

CONCLUDING REMARKS

In the previous section, we have shown a brief example of how we can talk about the discursive activity of a teacher considering her emotional engagement in it. We have also proposed an example of how rationality and emotion are entangled in teaching, pointing out how to integrate the analysis of what happens in the classroom with taking into consideration the fact that in decision-making actions are belief-related. The emotional orientation of the teacher allows us to talk about her beliefs when they are generating actions that are oriented towards reaching goals and understanding. The rationality and emotional orientation of the teacher are inseparable as the weave and warp of a fabric. The weave and the warp together shape the fabric in the same way as rationality and emotion together characterize the teacher. The warp is related to the emotional orientation as well as the weave is related to the actions. If we look at the backwards of the fabric, we find all prior experience of the teacher, without which she would not be the teacher she is now, with her beliefs and background. The interview is methodologically relevant with this respect: it is a means to investigate prior experience, looking for expectations that can constitute emotional orientations.

Our study also points to the entanglement of emotion and rationality as a possible way to rethink *intentionality*. Habermas argues that all action is intentional, yet he is not interested in treating emotion when dealing with the intentional and reflective character of rational behaviour. We suggest instead that for grasping the teacher's intentionality we need to consider her *emotional being* in decision-making, by virtue of the association of the latter with reflection on personal activity.

PERSPECTIVES ON THE USE OF HABERMAS' CONSTRUCT IN TEACHER EDUCATION: TASK DESIGN FOR THE CULTURAL ANALYSIS OF THE CONTENT TO BE TAUGHT

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This contribution presents some recent advances in our use of Habermas' construct of rationality in pre-service teacher education. The construct of rationality is adapted and made suitable to describe different forms of rationality in different mathematical domains; such a description is shown to be efficient to guide the planning and implementation of teacher education tasks. Furthermore, the construct of rationality may be progressively acquired by prospective teachers as a theoretical tool for their a priori analysis of classroom tasks and for the analysis of students' productions. As such, the construct of rationality may work as a tool both for teacher educators (guiding their design of a teacher education sequence) and for teachers (when they perform the Cultural Analysis of the Content to be taught—CAC).

The contribution may be situated in the stream of research on task design for teacher education (Watson & Mason, 2007); it may be linked also to the crucial issue of theories as tools for teachers, see Tsamir (2008).

In the first part of this contribution, the idea of CAC (Boero & Guala, 2008), as an important competence to be developed in teacher education, is briefly presented, together with the need of choosing suitable mathematical domains and tasks for it. In the second part, some examples of a-priori analyses of the same tasks tackled in different mathematical domains, performed according to the three dimensions of rationality, show how we elaborated the design of teacher education activities that may generate occasions for CAC. More specifically, we will deal with the design of teacher education activities that on the long term should enable teachers themselves to identify different forms of rationality, typical of different mathematical domains. Focus will be on the a-priori analyses of the first two tasks, then we will outline the whole teacher education sequence.

THE CULTURAL ANALYSIS OF THE CONTENT TO BE TAUGHT (CAC)

Boero & Guala (2008) present and discuss the Cultural Analysis of the Content to be taught (CAC) as one of the most important goals of teacher education. In the authors' words, CAC goes beyond Mathematics teacher professional knowledge, as described by Shulman (1986), "by including the understanding of how mathematics can be arranged in different ways according to different needs and historical or social circumstances, and how it enters human culture in interaction with other cultural domains" (p. 223). Moreover, the authors point out that CAC can help the teacher to reveal the nature of some difficulties met by students "as related to didactical obstacles inherent in the ways of presenting a given content in school, or to epistemological obstacles inherent in its very nature"(p. 226).

A crucial issue is how to promote CAC during teacher education programs. Boero and Guala point out that it is important to select suitable mathematical subjects and involve teachers in suitable activities. Nevertheless, "exemplary CAC activities on well-chosen topics can have an effect on other topics, if teacher education challenges teachers to reflect on those experiences and their cultural meaning beyond the specific content" (p. 229). In this way, well-chosen topics may serve a broader aim, overcoming the limitation of the specific subject at stake. The ideal routine of a teacher education program in a CAC perspective encompasses: individual problem solving, guided discussion of individual solutions (selected by the teacher educator), individual analysis of the given task, collective discussion in which the teacher educator acts as a mediator and offers some elements of CAC. Additional activities are the creation of tasks for students (individual creation and collective discussion).

CAC AND RATIONALITY

In a former contribution to PME (Boero, Guala & Morselli, 2013) the issue of different rationalities in different mathematical domains was presented. The working hypothesis of the present contribution is that performing activities across different mathematical domains, and promoting a reflection on different rationalities at issue in those domains, may be a major component of teacher education in a CAC perspective. In particular we will consider the case of the differences between synthetic geometry and analytic geometry rationalities: they can be traced back to the history of mathematics, thus promoting a view of mathematics as dynamic and cultural product. Furthermore, forcing teachers to solve a problem in different ways according to those different rationalities may challenge their beliefs about "closed" mathematical domains and put into evidence the possibility of having multiple solutions for the same problem by crossing the borders between different domains.

THE TEACHER EDUCATION ACTIVITY

The contribution refers to a prospective secondary teacher education course carried out in 2013 at the University of Genoa. 12 prospective teachers were proposed the following task to be solved individually:

The parabola task

a) To characterize analytically the set ***P*** of (non degenerated) parabolas with symmetry axis parallel to the ordinate axis, and tangent to the straight line $y=x+1$ in the point (1,2).

b) To establish for which points of the plane does it exist one and only one parabola belonging to the set ***P***.

c) To find straight lines that are parallel to the ordinate axis and are not symmetry axes of parabolas belonging to the set ***P***.

In terms of a-priori analysis, we may say that part (a) is formulated in a way that addresses students towards a solution with an analytic geometry method, or—eventually—a calculus method (but also a third method, referring to synthetic geometry considerations, might be possible).

Let us consider the set of parabolas through the point *(1,2)*: $y=ax^2+bx+2-a-b$.

To get the expression for those parabolas that are tangent to $y=x+1$ in (1,2):

> ***analytic geometry method***: you need to find relationships between a and b, such that the intersection points between $y=x+1$ and the parabola (which depends on a and b) are coincident in (1,2); after the system between the equation of the generic parabola through (1,2) and the equation of the straight line $y=x+1$, you substitute $y=x+1$ in the equation of the parabola, then you move to consider the condition of coincidence of solutions: discriminant $\Delta=0$, and you get $b=1-2a$.
>
> ***calculus method***: it is based on the meaning of the derivative $f'(c)$ as the slope of the tangent line in the point $(c, f(c))$; thus you get $f'(1) = 2a+b$ and you write $2a+b= 1$ (slope of the straight line $y=x+1$) . Finally you get $b=1-2a$.

The different methods encompass different forms of rationality, as already outlined in (Boero, Guala & Morselli, 2013). Here we consider a brief account of the rationality inherent in the solution of the part a) of the task with the analytic geometry method:

Based on the assumption that the solution of the problem is represented "within" the system, the solution may be made explicit by deriving an equation from the system and then getting the relationship between a and b through the analysis of that equation (Teleological Rationality, TR). Controls need to be performed on the different steps of the process (steps of algebraic transformations, algebraic substitutions, steps of the treatment of the equation $\Delta=0$, and so on) (Epistemic Rationality, ER). Effective communication requires the use of both algebraic and geometric language (Communicative Rationality, CR). We may observe how, apart from communication, verbal language plays a planning (TR) and control (ER) role, while algebraic language plays a prevailing executive role (TR).

Part (b) will be an object of work during the Research Forum session.

As regards part (c) (*To find straight lines that are parallel to the ordinate axis and are not symmetry axes of parabolas belonging to the set* ***P***), we may note that the problem may be solved easily by means of a synthetic geometry method:

> Among all the straight vertical lines, the only line that cannot be symmetry axis of a parabola of the set ***P*** is the line passing through (1,2) because, if it were symmetry axis, the vertex of the parabola (the point (1,2), would be on that line and the tangent line in that point would be horizontal, against the fact that the tangent line has slope 1.

Here again we may see how an analysis according to the components of rationality may be performed: Through a visual exploration of the problem situation (TR) the conjecture of the exclusion of the vertical straight line through (1,2) is produced; the validation of the conjecture (ER) is performed by combining (TR) visual evidence related to the shape of the parabola with the hypothesis about the slope of the given straight line (that is not horizontal) and getting a contradiction (ER). Communication

(CR) is based on the verbal and iconic language of geometry, with a narrative role for verbal language, together with some easy algebraic expressions to communicate technical details. Verbal language plays also a treatment role.

On the contrary, solving part C of the problem by means of analytic geometry looks much more difficult.

> x=1-(1/2a) is the equation of the symmetry axis of a parabola of the set ***P***. The exclusion of the line x=1 derives from the fact that 1=1-(1/2a) would imply the infinity of a. This solution is difficult to attain because it is necessary to explore the relationships between x and a in the equation x=1-(1/2a).

In this case, the solving strategy (TR) is based on the algebraic modelling of the situation, and on the interpretation of an algebraic equation; ER consists in the control of the different steps of the modelling process (algebraic formalization; and interpretation of the relationships between x and a); CR needs a narration of the process and a good technical verbal presentation of the discussion of the algebraic equation. Verbal language and algebraic language play a double role of communication (CR), and of treatment (TR). Control role (ER) is mainly played by verbal language.

The exemplified a priori analyses suggested that the task was a promising task for pre-service teacher education in a CAC perspective, since prospective teachers may appreciate the fact that different solutions are possible for the same task and that one can move from a problem formulated in a domain, to a solving process in another domain (with different rationalities).

Prospective teachers met big difficulties to solve parts (b) and (c) of the task. In the part (b), analytic geometry methods brought to several mistakes in performing rather complex algebraic transformations, with deadlocks depending on results "*without any meaning*", as a prospective teacher said. No prospective teacher tried to solve (b) through an easier synthetic geometry method. Interestingly, most teachers were unable to check their solutions through synthetic geometry considerations. A few teachers solved (c) by a synthetic geometry method, while most of those who tried an analytic geometry solution were unable to get the correct solution (and did not move to another method).

After working individually on the task, in a subsequent session prospective teachers analysed their own solving processes with the teacher educator. A special care was devoted to the analysis of the difficulties they met during the solving process, and to the difficulties met by some prospective teachers in the previous year. Indeed qualitatively similar data had been collected when the same task had been solved in an entrance examination for graduate students in Mathematics, willing to become mathematics teachers (see Boero, Guala & Morselli, 2013).

As a second step, prospective teachers were proposed a modified task that "forced" them to solve the same problem in a given way:

In question b) [referring to the previous task], how is it possible to exclude the right line $y=x+1$ and the right line $x=1$, within synthetic geometry (i.e. with your knowledge on the shape and other geometric features of the parabola)?

In question c) [referring to the previous task], how is it possible to exclude the right line $x=1$, within synthetic geometry?

After the individual solution, the difficulties met by participants were discussed and exploited to promote prospective teachers' need for performing the cultural analysis of the content to be taught. In doing so, the adaptation of Habermas' construct to mathematics education purposes (Boero & Morselli, 2009) was explicitly proposed to prospective teachers as a tool to identify specific, different features of typical activities in different mathematical domains; then the construct was used by them to perform gradually more autonomous a-priori analyses of tasks and of the difficulties students may meet to cross the borders between different mathematical domains. In this way, the construct of rationality was not only a tool for the teacher educator: it became a tool for prospective teachers.

RESULTS

The contribution to this Research Forum refers to the use of Habermas' construct as a tool for teacher education aimed at promoting CAC; a consequent result is the idea of inserting the construct of rationality into the professional knowledge of teachers, as a tool for them to perform the CAC and design suitable tasks for students.

Habermas' construct of rationality was refined to describe the different forms of rationality in different mathematical domains. Such construct was used for the design of a sequence of tasks conceived in the CAC perspective. The construct was also proposed to teachers as a tool to identify and compare specific features of activities in synthetic geometry and in analytic geometry, and of synthetic and analytic methods.

During the Research Forum session, examples of teachers' solutions and their trials to analyse them according to rationality criteria will be provided and discussed.

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