MATHEMATICAL MODELING IN SCHOOL EDUCATION: MATHEMATICAL, COGNITIVE, CURRICULAR, INSTRUCTIONAL, AND TEACHER EDUCATION PERSPECTIVES

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ABOUT THE RESEARCH FORUM

The purpose of this Research Forum is to present and discuss five perspectives on research and practice in the teaching and learning of mathematical modeling in K-12 school mathematics classrooms and to engage participants in advancing our understanding of the teaching and learning of mathematical modeling.

In today's dynamic, digital society, mathematics is an integral and essential component of investigation in disciplines such as biology, medicine, the social sciences, business, advanced design, climate, finance, advanced materials, and many more (National Research Council, 2013). In each of these areas, this work demands an understanding of and facility with mathematical modeling to make sense of related phenomena. Mathematics education is beginning to reflect the increased emphasis of mathematical modeling. In fact, mathematical modeling has been explicitly included in national curriculum standards in various countries. For example, in the United States, real-world applications and modeling are recurring features throughout the Common Core State Standards for Mathematics (CCSSM; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

In the past several decades, the mathematics education research community has made great efforts to study the issues related to the teaching and learning of mathematical modeling (Blum & Niss, 1991, Galbraith et al., 2007; Houston, 2009). Recent interest in mathematical modeling has been stimulated by OECD's PISA study, which assessed students' mathematical literacy, as well as the publication of the CCSSM in the United States. However, despite the increased interest in mathematical modeling, a large number of questions remain unanswered (see, e.g., Lesh & Fennewald, 2013). Blum (1994) pointed out “a substantial gap between the forefront of research and development in mathematics education, on the one hand, and the mainstream of mathematics instruction, on the other.” (p. 7). Twenty years later, this gap still exists (Kaiser, 2013). The main goal of this forum is to help narrow this gap with respect to the important area of mathematical modeling. In particular, this Research Forum provides a venue for researchers around the world to present findings and discuss issues surrounding the teaching and learning of mathematical modeling from the following five perspectives: Mathematical, Cognitive, Curricular, Instructional, and Teacher Education Perspectives. In each perspective, we list a set of research questions to be discussed.

MULTIPLE PERSPECTIVES FOR RESEARCH ON MATHEMATICAL MODELING: RESEARCH QUESTIONS

In this section, we first identify a few research questions in each perspective. In the next section, we provide some initial thoughts on some of the research questions.

Mathematical Perspective

The world of mathematics and the world of mathematics education interact, but do not completely overlap when they communicate with each other about mathematical modeling (Burkhardt, 2006; Pollak, 2003). Taking a parallel example, research on mathematical proof has shown that students and teachers hold different conceptions from those held by research mathematicians (e.g., Weber, 2008). Similarly, the notion of mathematical modeling in school mathematics is different from the way it is understood by practicing mathematical modelers. In fact, Lesh and Fennewald (2013) pointed out that one of the major challenges in the teaching and learning of mathematical modeling is the “conceptual fuzziness” about what counts as a modeling activity. Even those researchers who have long been conducting research on mathematical modeling have not come to an agreement on the processes of modeling and how to conceptualize mathematical modeling (Zawojewski, 2013). In this Research Forum, we specifically invite mathematicians and mathematics educators to directly interact and discuss these research questions about mathematical modeling. (1) *If we view mathematical modeling as a bidirectional process of translating between the real-worldand mathematics, what are its essential features?* (2) *Which of those essential features differentiate mathematical modeling from problem solving in school mathematics?* (3) *From the viewpoint of a practitioner of mathematical modeling, what are the essential competencies and habits of mind that must be developed in students to allow them to become competent mathematical modelers?*

Cognitive Perspective

In order to improve students' learning, it is necessary to understand the developmental status of their thinking and reasoning. Teachers' knowledge of students' thinking has a substantial impact on their classroom instruction, and hence, upon students' learning (e.g., Hill et al., 2007). Although we know a great deal about the cognitive processes of students' mathematical problem solving (see. e.g., Schoenfeld, 1992), we know less about how students approach modeling problems (Borromeo Ferri, 2006). Some researchers have theorized that students hold mental models that connect mathematics and the real-world(Borromeo Ferri, 2006). Even though there is little agreement about the fundamental cognitive features of mathematical modeling, there is some consensus that the *process* of getting from a problem outside of mathematics to its mathematical formulation in mathematical modeling begins with the formulation of research questions (Pollak, 2003). Prior research has demonstrated that students are quite capable of posing mathematical problems from given situations (Cai et al., in press; Silver, 1994), but it less clear how students formulate mathematical problems based on true real-worldsituations. It is important to note that the situations that have been used in problem-posing research are typically much less complex than the situations that occur in mathematical modeling. Hence, there is still much to learn from the cognitive perspective on mathematical modeling. (4) *What are factors that have an impact on students' formulation of researchable questions in modeling situations?* (5) *If we view mathematical modeling as ill-structured problem solving, how does one convert an ill-structured problem into a well-structured problem with specified research questions?* (6) *What are cognitive differences between expert modelers and novice modelers?*

Curricular Perspective

Historically, worldwide, changing the curriculum has been viewed and used as an effective way to change classroom practice and to influence student learning to meet the needs of an ever-changing world (Cai & Howson, 2013). In fact, curriculum has been called a change agent for educational reform (Ball & Cohen, 1996) and the school mathematics curriculum remains a central issue in our efforts to improve students' learning. Although some ideas fundamental to mathematical modeling have permeated school mathematics textbooks for some time (e.g., *Realistic Mathematics* in the Netherlands and *Standards-based mathematics curricula* in the United States), mathematical modeling is usually not a separate course, nor do there exist separate textbooks for mathematical modeling. Thus it will be useful to understand international perspectives on research questions from the curricular perspective. (7) *Looking within existing mathematics textbooks, are there activities specifically geared toward mathematical modeling?* (8) *Is it possible or even desirable to identify a core curriculum in mathematical modeling within the general mathematical curriculum?* (9) *In CCSSM in the United States, mathematical modeling is not a* *separate conceptual category. Instead, it is a theme that cuts across all conceptual categories. Given this orientation, how might mathematical modeling be integrated into textbooks throughout the curriculum?*

Instructional Perspective

Although curricula can provide students with opportunities to learn mathematical modeling, classroom instruction is arguably the most important influence on what students actually learn about modeling. Thus, the success of efforts for students to learn mathematical modeling rests largely on the quality of instruction that might foster such learning. Researchers have documented a number of cases of teaching mathematical modeling in classrooms (e.g., Lesh & Fennewald, 2013). In this Research Forum, we synthesize and discuss these findings to explore the following research questions: (10) *What does classroom instruction look like when students are engaged in mathematical modeling activities?* (11) *What mathematical-modeling tasks have been used in classrooms, and what are the factors that have an impact on the implementation of those tasks in classrooms?* In addition to devoting an appropriate amount of time to mathematical modeling tasks, teachers must also decide what aspects of a task to highlight, how to organize and orchestrate the work of the students, what questions to ask to challenge those with varied levels of expertise, and how to support students without taking over the process of thinking for them, and thus eliminating the challenge (NCTM, 2000). Subsequently, there is a need to consider how productive discussions around modeling activities can be facilitated. (12) *What is the nature of classroom discourse that supports students in becoming successful mathematical modelers*?

Teacher Education Perspective

There is no doubt that teachers play an important role in fostering students' learning of mathematical modeling and students' learning of mathematics through engagement in mathematical modeling. However, it is well documented that modeling is quite difficult for teachers because real-worldknowledge about the context for modeling is needed, and because teaching becomes more open and less predictable when students engage in more open-ended modeling situations (e.g., Freudenthal, 1973). In general, teachers' initial and in-service training as well as the curricular contexts of schooling have not readily provided opportunities to make mathematical modeling an integral part of daily lessons (Zbiek & Conner, 2006). A number of researchers in different countries (e.g., Kaiser & Schwarz, 2006) have started to develop mathematical modeling courses for in-service teachers. Likewise, a number of teacher education programs around the globe have included mathematical modeling as part of their initial teacher education program requirements (Galbraith et al., 2007). In this Research Forum, we discuss the various course offerings for teachers around the globe and address key research questions. (13) *Are there programs worldwide which successfully support pre-service and in-service teachers to teach mathematical modeling, and what are the features of these successful* *programs?* (14) *What level of familiarity with disciplines other than mathematics is it necessary for pre-service and in-service teachers to have in order to successfully teach mathematical modeling?*

MULTIPLE PERSPECTIVES FOR RESEARCH ON MATHEMATICAL MODELING: SOME INITIAL THOUGHTS

INITIAL THOUGHTS ON THE MATHEMATICAL PERSPECTIVE

This sub-section was written by John A. Pelesko, an applied mathematician. It presents a first-person perspective that represents a direct form of communication of ideas about mathematical modeling between an applied mathematician and mathematics educators.

Having spent the better part of the last twenty-five years engaged in teaching and doing mathematical modeling as an applied mathematician (see, e.g., Pelesko & Bernstein, 2003; Pelesko, Cai, & Rossi, 2013), it is hard to overstate the joy I felt upon realizing that the Common Core State Standards for Mathematics (NGA & CCSSO, 2010), the new standards adopted widely across the United States, , placed a special emphasis on mathematical modeling. This ascension can be credited, in part, to the long term efforts of researchers such as Pollak (2003, 2012), Lesh and Doerr (2003), and others who have argued that it is not just applications of mathematics that should be incorporated into the mathematics curriculum at all levels of education, but that the practice of mathematical modeling itself is an essential skill that all students should learn in order to be able to think mathematically in their daily lives, as citizens, and in the workplace (see, e.g., Pollak, 2003). Now that the importance of mathematical modeling is being recognized by the mathematics education community at large, appearing as both a conceptual category and a Standard for Mathematical Practice in CCSSM, it is critical that those who do mathematical modeling engage deeply with the K-12 mathematics education community around the issues of teaching and learning the practice. It is important to note that mathematical modeling is practiced far and wide – across the natural sciences, engineering, business, economics, the social sciences, and in almost every area of study in one form or another. Hence, the set of stakeholders in this conversation is large, and we should be careful not to substitute any one practitioner's perspective for the whole. Nevertheless, in an attempt to contribute to this conversation, here I provide one practitioner's perspective.

What is Mathematical Modeling?

Given the lack of attention that has been paid to mathematical modeling in the US educational system, especially in mathematics teacher education programs (see Newton et al., 2014), it is not hard to imagine that many mathematics educators, upon reading the CCSSM, found themselves asking this question. The brief description of mathematical modeling found in the standards document (pp. 72-73), and the fact that this description appears only within the high school standards, likely adds to this confusion. Further confusion is likely to occur as educators digest the US Next Generation Science Standards (NGSS Lead States, 2013), which make use of the term “model” both in and out of the context of “mathematical model.”

To address the question “What is mathematical modeling?” it is then perhaps useful to first consider the question “What is modeling?” My answer? Modeling is the art or the process of constructing models of a system that exists as part of reality. By “model,” I mean a representation of the thing that is not the thing in and of itself. The model captures, simulates, or represents selected features or behaviors of the thing without being the thing. By “mathematical model” I mean a model or a representation that is constructed purely from mathematical objects. So, mathematical modeling is the art or process of constructing a mathematical model. *That is, mathematical modeling is the art or process of constructing a mathematical representation of reality that captures, simulates, or represents selected features or behaviors of that aspect of reality being modeled.*

Now, we should note that mathematical models have a special place in the hierarchy of models in that they have both *predictive* and *epistemological* value. The epistemological value is a consequence of the idea that *mathematical modeling is a way of knowing*. The predictive value of a mathematical model gives mathematical models a special place in “science,” loosely and broadly defined, in that a mathematical model can take the place of direct ways of knowing, in other words, experiment. A good mathematical model is both an instrument, like a microscope or a telescope, allowing us to see things previously hidden, *and* a predictive tool allowing us to understand what we will see next.

Note that an especially “good” mathematical model, that is, one with a high level of predictive success, often ceases to be thought of as “just a model.” Rather, it attains a different status in the scientific community. We don't say “Newton's mathematical model of mechanics;” rather we say “Newton's Laws.” We don't say “Schrodinger's model of the subatomic world;” rather we say “Quantum Mechanics” or the “Schrödinger Equation.” Yet, each of these examples is, in fact, a mathematical model of the thing, and not the thing in and of itself. These examples have attained the highest possible level of epistemological value. They have become *the* way of knowing, understanding, describing, and talking about their subjects.

Now, we have diverged into abstract territory, and we do not want to leave the reader with the impression that mathematical modeling is *hard,* something to be left to the Newtons and Schrödingers of the world. Rather, we hope the reader is left with the impression that mathematical modeling is *exceedingly useful* and that by helping our students master this practice, we will be adding a tool to their mental toolkit that will serve them well, no matter what their future plans.

Thought Tools for Modeling

The question then becomes: How exactly does someone become a proficient mathematical modeler? In the United States, as evidenced by textbook after textbook on mathematical modeling (see, e.g., Pelesko & Bernstein, 2003), the answer has been "Modeling can't be taught, it can only be caught." Now, I take a different perspective and argue that it is useful to think of the mathematical modeler as having discrete "thought tools," each of which can be discovered and taught. As a consequence, we see that many "modeling cycles" unintentionally hide much of the real work of mathematical modeling.

We borrow the term "thought tools" and this framework for meta-thinking from the philosopher and cognitive scientist, Daniel Dennett. In Dennett (2013) he quoted his students as having made the observation that "*Just as you cannot do much carpentry with your bare hands, there is not much thinking you can do with your bare brain*." Dennett then proceeded by analogy with saws, hammers, and screwdrivers, to introduce thought tools of informal logic such as reductio ad absurdum, Occam's razor, and Sturgeon's Law. Applying this notion of thought tools to the mathematical modeler, we argue that they must possess a set of thought tools that lie in three different categories: Mathematical Thought Tools, Observational Thought Tools, and Translational Thought Tools.

Mathematical Thought Tools are those tools we attempt to add to our students' toolkits when we teach mathematics. These include notions such as algebraic thinking, the principle of induction, the pigeonhole principle, and any tool that lets students think about and do mathematics. Note that these thought tools are directed at mathematics and their utility is generally tied to thinking in the mathematical domain.

Observational Thought Tools are those tools we typically think of as being used by "scientists." These include the ability to think in terms of cause and effect, to observe spatial and temporal patterns in the real world, and to look deeply at reality. Note that these thought tools are directed at the real-worldand their utility is generally tied to thinking in the domain of the real world.

Translational Thought Tools are those tools that allow the mathematical modeler to take questions formed in the observational domain, translate them into the mathematical domain, and translate answers and new questions uncovered in the mathematical domain back again to the observational domain. These include knowledge of conservation laws, physical laws, and the assumptions that must be made about reality in order to formulate a mathematical model. Note that these thought tools are directed both toward reality and toward mathematics. Their utility lies in their usefulness in translating between these two domains.

In a typical "modeling cycle," such as appears in the CCSSM (see Figure 1), one moves from the "real world" or the "problem" to the "formulation" via a single small arrow. Buried in this small arrow is the use of Observational and Translational

Thought Tools. The remainder of the cycle, up to the point of comparing results with reality, generally relies purely upon Mathematical Thought Tools. While we can argue over whether or not we are properly equipping our students with the proper Mathematical Thought Tools they will need in their journeys around the modeling cycle, I would argue that generally we pay little attention to the Observational and Translational Thought Tools they will need to even begin their journey. Identifying, unpacking, and learning how to equip our students with these sets of tools is an essential step in learning how to teach mathematical modeling.

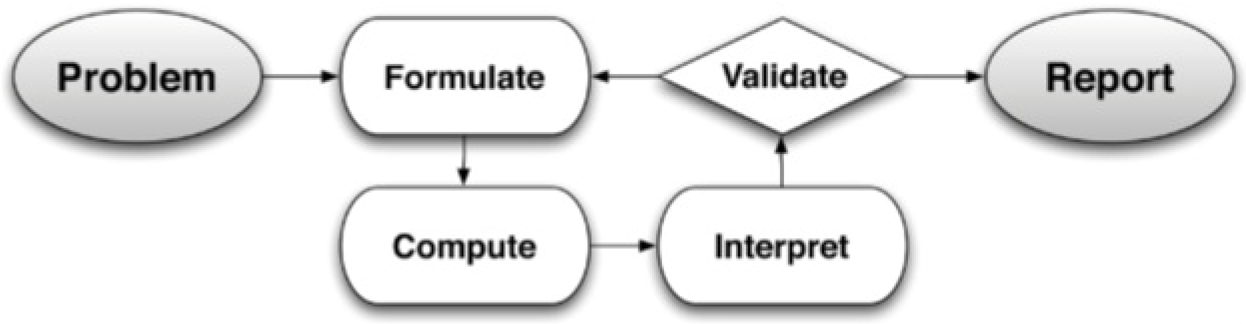

Figure 1: The mathematical modeling cycle from CCSSM (2010, p. 72)

As an example of how the mathematical modeler wields these tools, I ask the reader to imagine drops of morning dew on a spider web. Scientists, using their observational tools, notice these droplets and wonder why they are all roughly the same size. The mathematical modeler recalls that nature acts economically and often in a way that minimizes some quantity. They cast forth a hypothesis that here, nature is acting to minimize surface area, and that this leads the dew to break into droplets of nearly uniform size. They recast this observation and hypothesis into mathematical terms, already anticipating the mathematics from the presence of the notion of "minimizes" and wields their Mathematical Thought Tools to predict the size of the droplets given the presence of the dew. Comparing the predicted size with the size of actual droplets, the modeler refines and perfects the model, and acquires an understanding of *any* droplets on *any* spider web at *any* point in time.

In summary, mathematical modeling is a practice worth sharing and teaching. It is a powerful way of knowing the world, and it can be taught rather than simply caught. In the United States, we have much work to do in order to bring this new toolkit to our students. It will take the efforts not only of mathematics educators and applied mathematicians, but of mathematical modelers of every stripe in order to do so. Here, I have sketched out one avenue of approach that in many ways parallels recent work in unpacking the thought processes behind mathematical proof (see Cirillo, 2014). A similar effort to identify and unpack the thought tools of the mathematical modeler holds the promise of helping us train a wide range of students in the art of mathematical modeling.

INITIAL THOUGHTS ON COGNITIVE PERSPECTIVE

This sub-section was written by Rita Borromeo Ferri and Lyn English.

Past and Present

Cognitive perspectives on students' learning from modeling have long been debated within the international community. Nearly thirty years ago, Treilibs (1979) and Treilibs, Burkhardt and Low (1980) from the Shell Centre in Nottingham analysed, at a micro level, the videotaped modeling processes of groups of university students. They mainly focussed on determining how learners build a model and hence concentrated on the so-called “formulation phase”. They visualized this construction process of a model with “flowcharts” through which several modeling steps of individuals were represented graphically. One central result of their study was that building a model is a very complex activity for individuals and, at the same time, not easy to communicate for university professors during lectures. Because they only investigated university students, there was no empirical evidence about cognitive processes of primary, middle-school, or high school students. Unfortunately this group from Shell-Centre did not work on further studies.

Matos’ and Carreira’s (1995, 1997) research 15 years later placed a special emphasis on 10th-grade learners’ cognitive processes and representations while solving realistic tasks. They analyzed the creation of conceptual models (interpretations) of a given situation and the transfer of this real situation into mathematics. The results of their study show the numerous and diverse interpretations learners use while modeling and that the modeling process is not linear. Similar to the studies of Treilibs, Burkhardt and Low (1980), the research of Matos and Carreira did not emphasize the analysis of the complete modeling process.

Galbraith and Stillman (2006) also stressed cognitive aspects. They tried to identify the “blockages” that fourteen- and fifteen-year-old students experience while modeling, and pointed out that the overall modeling process is cyclic rather than linear. On the basis of their in-depth analysis, Galbraith and Stillman were able to identify in which parts of the modeling cycle individuals have blockages that hinder solutions. Their more recent research (e.g., Stillman, 2011) shows the important role of meta-cognitive activities while modeling, as does the research of Mousoulides and English (2008), which we address later.

Other significant research on cognitive perspectives includes the extensive work of Richard Lesh and his colleagues (cf. amongst others, Lesh & Doerr, 2003). They adopted a theoretical approach drawing upon upon the ideas of Piaget (1978) and Vygotsky (1934).

Also worthy of notice is the project DISUM (Blum & Leiss, 2007), which focused on the investigation of modeling processes of middle school students within a seven-step modeling cycle and on teacher interventions during these modeling activities. The results showed several micro-processes of students' work and how the situation model was built. The COM²-project (Borromeo Ferri, 2010) had a far stronger cognitive view than the project DISUM, with a focus on cognitive theory behind the analysis (Mathematical Thinking Styles). The central result of COM² was evidence of the reconstruction of "individual modeling routes" of pupils while undertaking modeling activities in the classroom. It became clear that mathematical thinking styles have a strong influence on the modeling behavior of students and teachers concerning their focus on "reality" and "mathematics" (Borromeo Ferri, 2011.

Summarizing some of the central research studies in this field, it becomes evident that cognitive views on modeling were highlighted in the international arena 30 years ago, but were then neglected for a long time and, in general, and were overtaken by other perspectives such as modeling competencies. However, the cognitive research increased especially after the ICMI-Study 14 on mathematical modeling, where the Discussion Document (Blum et al., 2002) argued that the cognitive psychological aspects of individuals during their modeling processes should be strongly emphasized in further studies.

The Cognitive Perspective – "An Additional Perspective"(?)

Kaiser and Sriraman (2006) offered a classification of five central perspectives on modeling, with a main focus on the goals intended for teaching modeling: realistic or applied modeling, contextual modeling (recently described as the "MEA-approach", Borromeo Ferri, 2013), educational modeling, socio-critical modeling, and epistemological modeling. These theoretical perspectives are understood as research perspectives. This classification was mainly a result of extensive discussions of international researchers during several European Conferences (ERME) within the group "Mathematical Modeling and Applications." As an additional perspective "cognitive modeling" or the cognitive perspective on modeling was formulated. Kaiser and Sriraman (2006) described "cognitive modeling" also as a "meta-perspective", because it is focusing on specific research aims and not on goals for teaching modeling, in contrast to the other approaches. When developing this classification, the general consensus was that this cognitive perspective can be combined with the other approaches depending on the research aims one likes to have in a study. Furthermore, Kaiser and Sriraman pointed out that the research aims of cognitive modeling are to describe and understand students' cognitive processes during modeling activities (Kaiser & Sriraman 2006).

Following the call from the ICME-14 Discussion Document, further research was done in the field of cognitive modeling. Results of empirical studies offered more knowledge about cognitive processes during modeling activities, especially concerning potential barriers or so-called red-flag situations (Stillman and Brown, 2011). When looking at the different modeling cycles (Borromeo Ferri, 2006), mostly a seven-step-modeling cycle (Blum & Leiss, 2007;Borromeo Ferri, 2006) is used as a basis or an instrument for analysing cognitive processes along several steps. Within the current discussion the seven-step-cycle is described as the "diagnostic modeling cycle" because this cycle includes the step, "construction of a situation model." Building a situation model or a mental representation of the situation is a very individual process, because one has to understand the problem and visualize the given situation (Blum & Leiß, 2010; Borromeo Ferri, 2010).

On the one hand there are a lot of studies that have a focus on theory-building, but on the other hand, we now have a lot of implications, core concepts, and empirical evidence that this cognitive view is no longer exclusively a research perspective or an "additional perspective" as described in the initial classification of Kaiser and Sriraman (2006). These researchers argued that the cognitive view on modeling is mostly integrated in empirical studies, because it is a crucial part of modeling activities. But we believe that this "additional perspective" is far more than a "meta-perspective" and should have an equal position to the other named perspectives.

Cognitive Modeling in School

Within the cognitive perspectives on modeling we give an additional characterisation of such perspectives on the basis of several studies done by Borromeo Ferri (e.g., 2007, p. 265): "If modeling is considered under a cognitive perspective the focus lies on the individual thinking processes which are expressed mainly through certain verbal and non-verbal actions in combination with written solutions during modeling activities of individuals (including teachers)."

A further example of this cognitive perspective can be found in the research of Mousoulides and English (2008). They reported on the mathematical developments of two classes of ten-year-old students in Cyprus and Australia as they worked on a complex modeling problem involving interpreting and dealing with multiple sets of data. The MEA problem ("The Aussie Lawnmower Problem") required students to analyse a real-worldbased situation, pose and test conjectures, and construct models that are generalizable and re-usable. Their findings revealed that students in both countries, with different cultural and educational backgrounds and inexperienced in modeling, were able to engage effectively with the problem and, furthermore, adopted similar approaches to model creation. The students progressed through a number of modeling cycles.

In the first cycle, the students focused only on some of the problem data and information. This resulted in a number of initial, interesting approaches to model development, but these approaches were inadequate because the students did not take into account the whole problem data. The students quickly moved to a second cycle when they realized that their initial approaches were not successful, since a number of contradictions arose in their results. Consequently, almost all groups in both countries moved to mathematizing their procedures by totalling the amounts in each given table of data and, for the Australian students, by finding the averages. This was a significant shift in the students' thinking. In the third cycle, the students in both countries identified trends and relationships to help them find a solution to the problem.

Also of significance in Mousoulides and English's (2008) study is students' engagement in self evaluation: groups in both countries were constantly questioning the validity of their solutions, and wondering about the representativeness of their models. This helped them progress from focusing on partial data to addressing all data in identifying trends and relationships in creating better models. Although the students did not progress to more advanced notions such as rate (which was beyond the curriculum level in both countries), they nevertheless displayed surprising sophistication in their mathematical thinking. The students' developments took place in the absence of any formal instruction and without any direct input from the classroom teachers during the working of the problem.

INITIAL THOUGHTS ON THE CURRICULAR PERSPECTIVE

This sub-section was written by Marcelo Borba and Geoffrey Wake.

As Cai and Howson (2013) pointed out in their discussion of "What is a Curriculum?" there is no agreement over a definition of the term. Taking curriculum to refer to intentions, it can be considered as formal documentation that sets out what is to be taught and learned and as such, it encapsulates an epistemology with historical precedence. However, as Travers and Westbury (1989) highlighted, it is possible to broaden consideration of the curriculum by not only focusing on what is *intended* but also what is *implemented* and what is *attained*. This removes mathematics from the pages of official documents and brings it to life in the schools and classrooms where it is taught (*implemented*) and learned (*attained*). It is in such classroom ecologies, or in the classroom milieu as Brousseau (1989) called it, that mathematics is lived and defined for students. Ultimately mathematics becomes something uniquely defined for each individual through the mathematical activity in which they take part, both socially and alone, although there are certainly common and strong trends that emerge in classrooms, schools and indeed nationally (Givvin et al., 2005). Taking a socio-cultural view, mathematics, its teaching and its learning, can be considered as in mutually recursive relationship with the classroom community in which teachers and students live and learn. It is in this coupling of human activity with mathematics as a discipline that modeling as a mathematical practice seeks to find a place.

Historically, worldwide, changing the intended curriculum through carefully designed (re-)specification has been viewed and used as an effective way to change classroom practice and to influence student learning to meet the needs of an ever-changing world (Cai & Howson, 2013). In fact, curriculum (intended and specified) has been called a change agent for educational reform (Ball & Cohen, 1996) and the school mathematics curriculum as such remains a central issue in our efforts to improve students' learning. Further, in terms of bridging from strategic and tactical design (Burkhardt, 2009) to classroom practice, through the technical design of classroom materials, we find little support. Although some ideas fundamental to mathematical modeling have permeated school mathematics textbooks for some time (e.g., *Realistic Mathematics* in the Netherlands and *Standards-based mathematics curricula* in the US), mathematical modeling is usually not a separate course, nor do there usually exist separate textbooks for mathematical modeling. Thus it will be useful to understand international perspectives based on research questions from the perspective of curriculum.

Discussion is further complicated if we consider different understandings of what modeling as a mathematical practice means and the different aspects of it. For instance, if one considers modeling in the classroom from the perspective of connecting mathematics to real-worldproblems or problems from everyday life, it is possible to think of changes in textbooks that can include these aspects. But if one considers modeling from a perspective in which the emphasis is on the choice of the problem by the students the situation may change. Borba and Villarreal (2005) see modeling as "a pedagogical approach that emphasizes students' choice of a problem to be investigated in the classroom. Students, therefore, play an active role in curriculum development instead of being just the recipients of tasks designed by others" (p.29).

In such an approach the curriculum is not pre-defined and specified, it is negotiated between teachers and students, and consequently the students' interests are a priority. The authors suggested that such an approach would approximate the practice of applied mathematicians, who deal with new issues, and in which one of the main tasks is "building the problem", defining the variables and then trying to solve the resulting mathematical model, usually under time pressure. João Frederico Meyer, an applied mathematician, in a book written with two mathematics educators, reinforces the idea that there is time pressure and that finding the problem is a big part of applied mathematics (Meyer, Caldeira and Malheiros, 2011). If this is the case, new questions may arise; for example, "Do we need to have a list of topics to be taught?" Taking such a view requires us to consider new directions in discussions about curriculum as it is intended, implemented and attained.

Authors such as Skovsmose (1994) also propose, and have done so for a long time, that modeling may (or should) be closely linked to social and political issues. He identifies critical mathematics education as being closely connected to modeling. In such a perspective, it is not so relevant that the choice of problems is made by the students, but it is important that the theme discussed in the classroom is closely connected to issues such as social equality and justice. We should perhaps also add other issues such as those relating to gender differences and the environment to reflect emerging concerns of citizens throughout the world. From such a perspective one can ask: is it possible to enroll students in political discussion, with capital P, if we have a problem that was chosen by the teacher or is from a textbook?

A long time ago, Borba (1990) asked similar questions when he connected ethnomathematics with modeling in informal education settings in one of the slums of Brazil. If ethnomathematics – with its concern with cultural background of students is brought into curriculum debate – is combined with modeling, then different issues and questions may arise, such as: (1) Can a common textbook be used with students from different backgrounds in different parts of a country? and (2) How do we deal with multicultural classrooms?

Borba (2009) and Meyer, Caldeira and Malheiros (2011) have debated the synergy between modeling and digital technologies. Authors such as these have discussed how modeling can be transformed with technology as students can be released from calculations and focus on problems that could not be handled if digital technologies were not available. Soares and Borba (2014) have shown how an inversion of topics can be made in an introductory Calculus course for Biology majors if software such as Modellus is available. It was found that such students could start, from day one, dealing with a modeling activity related to malaria, using a model that was important in the second half of the 20th century. This model included a system of differential equations with students computing graphical solutions and graphically displaying these using Modellus. Such a model was used with these students to introduce several concepts in precalculus and calculus, including the notion of differential equations by the end of the course. This approach shows a clear possibility of how inversion of the order of topics taught in the curriculum is possible due to the use of technology-based modeling tools. This leads to further potential research questions such as, “To what degree do students need to learn the formal mathematical techniques of differentiation and integration, for instance, when students are able to model with access to digital technologies?”

A further perspective we might explore focuses on modeling by workers in settings out of school. Most recently Wake (2014) has suggested how mathematics in general education might learn from activity in workplaces. In summarizing findings from some dozen case studies of the mathematical activity of workers he reported: “Workplace activity with mathematics as central often relies on relatively simple mathematics embedded in complex situations (Steen, 1990). Making sense of this also provokes breakdowns, problem solving and modeling” (Wake, 2014, p. #)

The complexity of the situations that workers deal with is considerable, but of course it is an integral part of their daily life, and consequently, in their work, they often do not recognize that what they are doing involves mathematics at all. It certainly seems to bear little resemblance to the mathematics they met in school. This raises the important question: How can we better provide experiences of modeling in school that ensures good preparation for activity of this type in out of school settings such as workplaces?

Wake went on to suggest one way that we might reframe mathematics curricula by suggesting a model that could support the didactical transpositions that Chevallard (2002) identified as necessary in adapting mathematical knowledge for use in the day-to-day interactions of mathematics classrooms. This recognized how we must attend to the design that is essential if we are to bring into reality our aims and values in relation to modeling. As we highlight here, there are many different perspectives that might inform approaches to developing appropriate mathematics curricula, and these raise many different potential research questions. It is clear that a comparative approach to such research would be beneficial by providing additional insight as we have increased opportunities to test our hypotheses in a range of different cultural settings. A starting point is to focus on curriculum intentions, but the real richness of such work will be revealed as we explore modeling activity in classrooms throughout our international community.

INITIAL THOUGHTS ON THE INSTRUCTIONAL PERSPECTIVE

This sub-section was written by Vincent Geiger and Gabriele Kaiser.

While instruction in mathematical modeling shares many of the characteristics of quality teaching and learning in mathematics. at the same time, it is inclusive of a range of practices that are not a part of the traditional mathematics classroom (Niss, Blum & Galbraith, 2007). Approaches to teaching modeling can involve traditional methods or be based on innovative teaching practices such as inquiry methods, collaborative group based learning, and use of digital technologies. The nature of instruction in mathematical modeling varies according to many factors including: level of education, national context, curriculum intention and expectation, type of modeling tasks, and availability of teaching resources. Modeling tasks on which instruction is based can be drawn from a range of real-life situations including industry and the workplace, social and political issues, or daily life. Different contexts have implications for the design of modeling tasks and the selection of associated pedagogies.

This paper provides a brief synthesis of selected aspects of instruction in mathematical modeling. In doing so, we consider types of modeling activities and tasks and approaches to mathematical modeling teaching practice.

Modeling Cycles, Activities, and Tasks

The process of mathematical modeling remains a source of debate within the mathematical modeling community. The dominant perspective depicts mathematical modeling as a cyclic process in which mathematics is brought to bear on real-worldproblems through a series of steps or phases. While various forms of the modeling cycle are described in the literature (e.g., Blum, 1995; Kaiser, 1996; Pollak, 1968), these typically coalesce around a number of core activities: central influencing factors are identified; the real problem is simplified in order to build a manageable model of the situation; assumptions based on known factors are made to accommodate missing information; the real situation is translated into an idealised mathematical model; an initial solution is generated from the mathematical model; proposed solutions are tested against the initial real-worldsituation; a decision is made about the validity of a solution; and the process is revisited until an acceptable solution is established. These phases can take place in a linear fashion or frequent switching between the different steps of the modeling cycles may occur in generating a final solution (Borromeo Ferri, 2011). The modeling of real-worldproblems is challenging and so students will typically experience blockages to their progress (e.g., Stillman and Galbraith, 2006). These blockages can be related to limitations in their content knowledge, cognitive impasses, and obstacles associated with beliefs or attitudes.

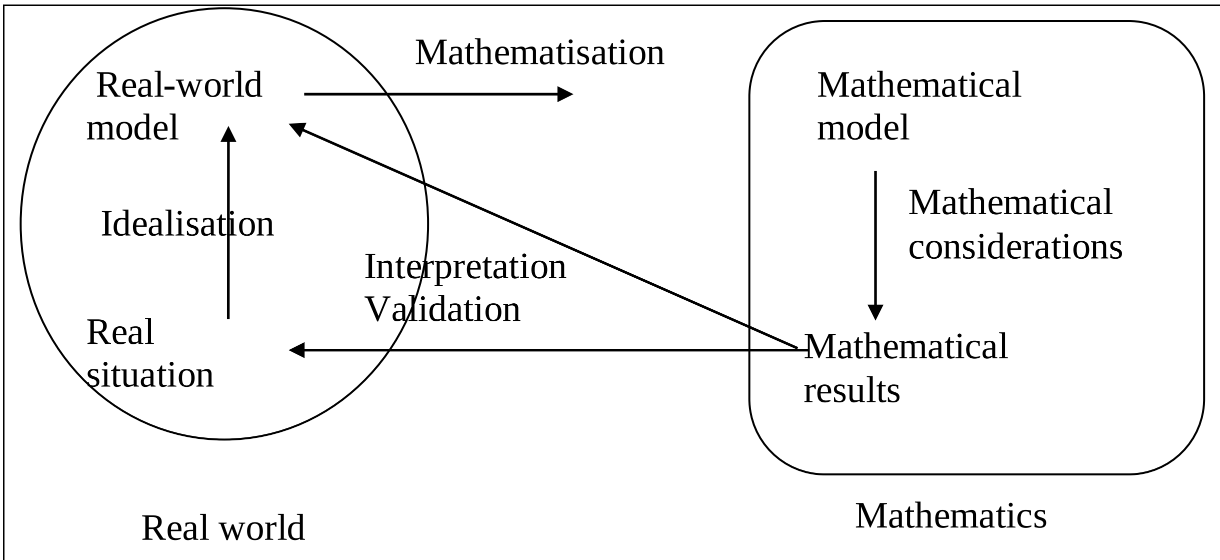

Fig. 2: Modeling process from Kaiser-Meßmer (1986) and Blum (1996)

Other modeling approaches place cognitive analyses in the foreground and so include an additional stage within the modeling process, the understanding of the situation by the students. In this approach students develop a situated model, which is then translated into the real model (Blum, 2011). This approach is represented in Figure 3.

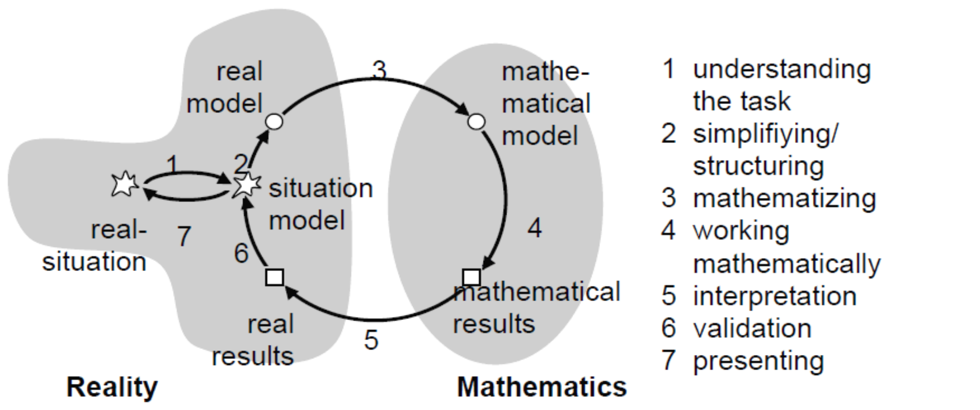

Fig. 3: Modeling process by Blum (2011)

While this cyclic process is consistent with the way many real-worldproblems are modeled, others argue for a broader definition for modeling that accommodates a wider range of context aligned mathematical activity. Modeling is considered by Doerr and English (2003), for example, as ‘‘systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behaviour of some other familiar system’’ (p. 112). From this perspective, modeling makes use of mathematical thinking within realistic situations to accomplish some purpose or goal but may or may not involve a cyclic process. Alternatively, Niss (2013) distinguished between descriptive and prescriptive types of modeling. In descriptive modeling a real-world problem is specified and idealized, assumptions are made, relevant questions are posed, leading to the mathematization of the problem. Answers are then derived and justified and de-mathematized and finally validated. Thus, the processes associated with descriptive modeling are consistent with the cyclic view of mathematical modeling. By contrast, the purpose of prescriptive modeling is not to explain or make predictions about real-worldphenomena but to organize or structure a situation, for example – where should a new power plant be located? As the nature of prescriptive modeling cannot involve the validation of an initial solution, the process is not cyclic. Thus, Niss’ insight into the nature of mathematical modeling suggests that the real-world phenomenon being investigated influences the way it is modeled, which in turn has implications for how instruction is organized to support students to work on a problem.

Approaches to Modeling Practice

The purpose of modeling from an instructional perspective can be considered as an objective in itself or as a method to achieve the goal of mathematics knowledge construction (Ikeda, 2013). The first purpose is based on the premise that the capacity to model and to find solutions to life related situations is a competence that can serve the individual in daily life and in the workplace. The second purpose is achieved when an individual constructs new knowledge or re-constructs knowledge they have already acquired (Van Den Heuvel-Panhuizen, 2003) when engaging with the process of modeling. As modeling requires the use of previously acquired mathematical knowledge in different ways it promotes a flexible and adaptable mindset in relation to the utilization of mathematical competencies. Challenging modeling problems, however, demand the appropriation of new mathematical facts, skills and processes, thus requiring the construction of new knowledge.

Niss and Blum (1991) distinguished six different approaches to instruction related to mathematical modeling and applications:

- separation – in which mathematics and modeling are separated in different courses;
- two-compartment – with pure and applied elements within the same course;
- islands – where small islands of applied mathematics can be found within the pure course;
- mixing – in which newly developed mathematical concepts and methods are activated towards applications and modeling, although the necessary mathematics is identified from the outset;
- mathematics curriculum integrated – here real-life problems are identified and the mathematics required to deal with them is accessed and developed subsequently;
- Interdisciplinary integrated – operates with a full integration between mathematics and extra-mathematical activities where mathematics is not organized as separate subject.

While these approaches to instruction in mathematical modeling are distinct, they should not be seen as mutually exclusive, but rather as a choice to be made by teachers that reflects their intention when planning for instruction. This choice will impact the way they design modeling tasks (e.g., Geiger & Redmond, 2013). The design of tasks is also framed by the affordances and constraints of educational systems and school based circumstances. Tasks can be extended complex modeling problems in co-operative, self-directed learning environments (e.g., Blomhøj & Hoff Kjeldsen, 2006) through to more constrained versions of modeling tasks embedded taught within a traditional curriculum (e.g., Chen, 2013).

The nature of modeling task design, however, becomes increasingly complex once digital technologies are introduced into the range of resources available to students and teachers. Research into the role of digital technologies in supporting mathematical modeling indicates that more complex modeling problems become accessible to students (Geiger, Faragher, & Goos, 2010), but the successful implementation of technology “active” modeling tasks is largely dependent on the expertise and confidence of teachers as well as their beliefs about the nature of mathematics learning.

INITIAL THOUGHTS ON TEACHER EDUCATION PERSPECTIVE

This sub-section was written by Gloria Stillman and OhNam Kwon.

Earlier in this document, it was pointed out that the teaching profession faces difficulties in teaching mathematical modeling as mathematical content in its own right and using mathematical modeling as a teaching strategy to engage students in the learning of mathematics. Further, this becomes problematic for many teachers because of the different practices teachers must employ or adopt associated with allowing students more freedom to drive their own learning and the amount of specific domain knowledge that might be required. García and Ruiz-Higueras (2011) suggested that this problematic issue can be viewed from the perspective of renewal of the profession as a whole thus taking a top-down approach in researching issues associated with it, or alternatively, as a problem of the teacher in the classroom in renewing their models of teaching leading to research that focuses on more of a bottom-up approach. Both of these approaches are evident in the research literature associated with research into teacher education related to teaching modeling, whether it be researching in-service or pre-service teachers. In this section we examine the extent to which such research has taken as its focus (a) programs that support pre-service and in-service teachers in teaching mathematical modeling, and (b) interdisciplinary or extra-mathematical knowledge requirements for successfully teaching mathematical modeling. We also suggest where there are current gaps and the implications for future research.

Nature of Research into Teacher Education in Modeling

Many of the reports of studies into teacher education with respect to modeling are small-scale qualitative research studies involving the reporting of rich data from a few teachers usually from case studies (e.g., Villareal, Esteley, & Mina, 2010). This can be seen as either a sign that the research field is emerging or of the complexity of the phenomenon being studied (Adler et al., 2005). Both are clearly true. A third possibility is the way research is predominately reported in the field. Much research in this area is reported in short conference papers (e.g., Ng et al., 2013; Widjaja, 2010) or short book chapters (e.g., Stillman & Brown, 2011) in edited research books, and authors might not see these as ideal contexts for reporting larger studies. The focus of this research is teachers in teacher preparation and in-service courses. We have not found any studies where the reported focus is the teacher educators themselves and their expertise in supporting the teaching profession to address modeling so this is an area for future research.

Researching Programs Supporting Pre-service and In-service Teachers in Teaching Mathematical Modeling

Several programs for supporting pre-service teachers to teach mathematical modeling have begun to be developed and described around the world (e.g., Biembengut, 2013; Hana et al., 2013; Kaiser & Schwarz, 2006; Kaiser et al., 2013). A common approach is to involve pre-service teachers in modeling activities in order to develop a connected knowledge base in mathematics of both skills and concepts that can be applied to a variety of phenomena. There has, however, been limited research of the effectiveness of such programs. Often, the research is more of an exploratory nature investigating how modeling experiences can be infused into existing programs (e.g., Widjaja 2010, 2013). Table 1 shows a small selection of studies with pre-service teachers (PSTs) as the focus and selected claims or findings from these. In-depth evaluation studies identifying the ingredients of successful programs that can be scaled up for large course offerings should be the focus of future research.

Program	Studies	Selected Findings/Claims
Brazillian PSTs	Biembengut (2013): study of course offerings across Brazil	Too little emphasis on MM although present in courses in all states; potential usefulness of MM developed in PSTs through such courses
Indonesian PSTs	Widjaja (2010); Widjaja (2013): study of MM activities	Must encourage PSTs to state assumptions & real-worldconsiderations of model in order to validate its appropriateness & utility
US elementary PSTs	Thomas & Hart (2010): models & modeling approach with Model Eliciting Activities (MEAs)	PSTs struggle with ambiguity of modeling activities; need to develop PSTs' ability to engage collaboratively with MEAs
Singaporean secondary mathematics PSTs	Tan & Ang (2013) using MM activities	PSTs need to experience MM for themselves developing meta-knowledge about modeling through such experiences
South African PSTs	Winter & Venkat (2013) using realistic word problems	PSTs abilities to reason within problem context critical; must develop deep, connected understanding of elementary mathematical content for successful modeling through such experiences

Table 1: Exemplar studies with pre-service teachers as focus.

PD Program/Course	Reports	Findings/Claims
LEMA (Learning and Education in	Schmidt (2012): Pre, post & follow-up	Motivations to include MM in teaching which increased after the

and through Modeling and Applications) 2006-9	questionnaire for participants in training course and a control group; supplemented by interviews	training course: Increases students ability to calculate & think more creatively, work independently & see relevance of mathematics to everyday life; modeling tasks have long term positive effects in mathematics lessons & beyond these and lesson teacher's workload
Making Mathematics More Meaningful M4	Berry (2010): design based research study	Refined group observation & teacher self-coaching tools designed & tested for teacher facilitation of optimizing student functioning in group work on MEAs
Experience 2004 with 3 secondary mathematics teachers & a university teacher	Villareal et al. (2010): main focus of report is student & task	MM offers space to construct new meaning for use of Information and Communication Technologies (ICTs) & ICTs are the media to think with and produce MM processes; Teacher, students & ICTs constituted a powerful thinking collective of Humans-with-Media
Training Program for non-certified teachers in Brazil	de Oliveira & Barbosa (2013)	Tensions in discourses can contribute to teacher PD through actions & strategies to deal with them; discussion of these tensions should be part of PST education
German in-service secondary mathematics teachers in academic-track schools	Kuntze (2011): quantitative comparative study of views	In-service teachers compared with PSTs saw a higher learning potential for tasks with higher modeling requirements; were less fearful of the inexactness of MM tasks; did not report good meta-knowledge about modeling.

Table 2: Exemplar studies with in-service teachers as focus.

In contrast, professional development (PD) programs or courses for in-service teachers have received much more research attention (e.g., de Oliveira & Barbosa, 2013) as these usually have been part of a funded project (e.g., LEMA see Table 2) of fixed duration with a research and evaluation study attached to it contingent on its successful completion in an expected time frame. Many results are localised to the context in which the programs were conducted but others clearly transcend contexts. Table 2 shows a small selection of studies with in-service teachers as the focus and selected claims or findings from these.

Researching Interdisciplinary or Extra-mathematical Knowledge Requirements for Successfully Teaching Mathematical Modeling

Within the studies of teacher education examined, there were few studies that addressed interdisciplinary or extra-mathematical knowledge requirements for successfully teaching mathematical modeling directly although some explained their findings (e.g., Tan & Ang, 2013; Winter & Venkat, 2013) by suggesting pre-service teachers isolated their modeling from the real-worldsituation in focus (e.g., car stopping distances), activated real-worldknowledge and attempted to incorporate such considerations into their modeling (Widjaja, 2013) or used contextual knowledge to interpret final mathematical answers (Winter & Venkat, 2013) within the problem context. Many classroom studies were found that alluded to the necessity for teachers, even in elementary settings, to have the knowledge background to make this knowledge visible to students. Mousoulides and English (2011), for example, when investigating the classroom activities of 12-year-old students exploring natural gas worldwide reserves and consumption, asked:

How we might assist students in better understanding how their mathematics and science learning in school relates to the solving of real problems outside the classroom and how we might broaden students' problem-solving experiences to promote creative and flexible use of mathematical ideas in interdisciplinary contexts?

They highlighted the issue of how the nature of engineering and engineering practice that relates to such problems can be made visible to these students. Studies which directly address interdisciplinary or extra-mathematical knowledge requirements for successfully teaching mathematical modeling are an area for future research.

CONCLUSION

This Research Forum starts to address a set of research questions in each perspective. Through the presentations and discussion, we hope to present a state of the art about the research on mathematical modelling from each perspective. After the conference, the organizers plan to develop a journal special issue and a book on the teaching and learning of mathematical modeling based on this Research Forum. We welcome all participants to contribute their ideas and papers.

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