THE CHALLENGES OF TEACHING MATHEMATICS WITH DIGITAL TECHNOLOGIES – THE EVOLVING ROLE OF THE TEACHER

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This Research Forum highlights the most recent research on the development of the role of the teacher of mathematics within mathematics classrooms that involve the use of technological tools, with an emphasis on teachers' experiences within both formal and informal professional development programmes. We foreground the theoretical ideas and methodological approaches that focus on the development of classroom practices at the levels of both individual teachers and communities of teachers, charting their respective development over time. The RF makes reference to a previous forum at PME37 on the theme of Meta-Didactical Transposition (Aldon et al. 2013a), a theoretical framework that has evolved from research in this area.

INTRODUCTION

The earlier research concerning digital technologies directed their lenses on the processes and outcomes of pupils' mathematical learning. However, it is now widely acknowledged that the earlier visions for how pupils' learning might be transformed by the inclusion of technology have not translated into widespread changes in classroom practices. This is partly due to an underdeveloped knowledge of how teachers' practices are impacted by the use new of technologies, and subsequently how teachers embed them within their professional lives, for the purpose of improving pupils' mathematical learning. More recent research has focused on the development of teachers' knowledge and practices within technology enhanced classroom environments. For example, the instrumental approach used in didactics of mathematics (Artigue 2002; Trouche 2005), initially used to analyse students' interactions with technology in mathematics learning, has been applied to the study of teachers' professional development through its central notion of "instrumental genesis", using the concept of orchestration and its extension (Drijvers et al. 2010; Trouche 2005). During PME37 the development of teacher's practices with technology has also been discussed extensively at a Research Forum on *Meta-Didactical Transposition (MDT)* (Aldon et al. 2013a; Arzarello et al. 2014). Other ways to describe the use and knowledge of technologies by teachers is given by theories such as *Pedagogic Technological Knowledge (PTK)* (Hong and Thomas 2006; Thomas and Hong 2005), *Technological, Pedagogical and Content Knowledge (TPACK) (Koehler and Mishra 2009; Mishra and Koehler 2006)*, and the *Structuring Features of Classroom Practice* framework (Ruthven 2009). A comprehensive discussion comparing *TPACK*, the *Structuring Features of Classroom Practice* Framework and the *Instrumental Orchestration Approach* can be found in (Ruthven 2014). Further to this, research on teacher identities has also contributed insights into how and why teachers develop their practice (or not) as users of digital technologies. From a sociocultural perspective, teachers' learning is conceptualised as the evolution of their participation in practices that develop their pedagogical identities, which Wenger describes as "a way of talking about how learning changes who we are" (1998).

As this Research Forum is focused on making visible the dynamic processes of teachers' development of their classroom practices with and through technology over time, the theoretical frameworks have been chosen as they enable this temporal element to be seen. However, our choices are not exhaustive!

The adoption of a holistic view of teachers, their practices and their professional learning concerning the teaching of mathematics with digital technology raises many questions (about practices, about training etc.). Newer constructs have been developed to articulate the teachers' learning processes with and about mathematical digital technologies, such as *critical incidents* (Aldon 2011), *hiccups* (Clark-Wilson 2010) and the notions of *instrumental distance* (Haspekian 2005) and *double instrumental genesis* (Haspekian 2011, 2006).

Many studies evidence the importance of the role of the teacher *from different perspectives*: the teacher in the classroom, the teacher as a learner of mathematics, the teacher as member of a community of professionals (Sfard 2005). For instance, Wenger (1998) argued that teachers have to reconcile multiple identities that result from their participation in various communities of practice into a single core identity that holds across contexts. Theorising teacher learning as identity development in multiple contexts provides a dynamic perspective on the evolution of teachers' knowledge and practices. This approach is useful for investigating how teachers engage with any kind of educational innovation, whether this involves the introduction of digital technologies, other teaching resources, or changes to curriculum or assessment.

As Wenger's theory suggests, any research into teaching practices must confront the issue that teaching practices embody several different dimensions (social, institutional, cognitive…). Consequently researchers have to make choices about the 'grain size' of their focus of analysis, to different levels of detail, whilst also respecting the dynamicity and interconnectivity of the related processes. The analysis appears to be even more complex when digital technologies are introduced, both as tools for teaching and as tools within teacher education. When the research lens is trained on the mathematics teacher in his/her interaction with the technology in the class and during professional development activities, it is a challenge to maintain a deep focus on multiple aspects.

This Research Forum aims to respond directly to this challenge. It is focused on the role of the mathematics teacher within both the classroom and during teacher education activities, where the mathematical, pedagogical and wider communication tools include increasingly ubiquitous digital technologies. The Forum aims to advance research on teaching practices in general by drawing from the substantial research of the last 5-10 years on teachers' uses of digital technologies in school mathematics in order to explore and propose stronger connections with the wider body of research on teachers' practices with technology and learning from cognitive, psychological, and social perspectives. The main objective is to contribute to a critical debate on the wider implications of the selected set of research themes on initial and continuing teacher education.

MAKING SENSE OF THE EVOLVING ROLE OF THE TEACHER

We start by postulating that the process through which teachers develop their professional identity and associated practices over time is experienced as 'professional development', which encompasses the full range of individual and collaborative activities in which a teacher might engage, within and outside of their school setting, to include: traditional courses; within-school initiatives; participation in research projects; and professional networks.

The term 'professional development' is being conceived as both a product (i.e. a tangible set of professional activities with structure, content, a timeline etc.) and as a process, which involves a range of participatory actions. This is analogous to the idea of a mathematical proof, where the final product can be conceived as the outcome whereas the process of proving may well have involved exploration, argumentation, justification, communication etc.

In order to analyse the process of mathematics teacher professional development holistically and from different theoretical perspectives, we have identified some key questions that address three axes of research concerning teachers' practices: the professional development of the individual teacher; the role of digital technological tools; and the role of institutions.

- How can we observe and describe change, evolution of practices and innovation within mathematics teachers' professional development concerning digital technologies?
- How does the use of digital technological tools impact upon the role of the teacher and their associated professional development?

- What roles do the institutions play (e.g. national curriculum, national/international assessment, school inspection regimes, etc.) in supporting changes within mathematics teachers' professional development at large scale?

Different theories that try to describe the activity of teaching involve different dimensions. In order to address our key-questions we have identified among these dimensions five themes that include a consideration of the process of professional development concerning digital technologies:

- The institutional context and its impact upon teachers' roles.
- The design of selected mathematics teachers' professional development programmes (from the perspective of the designers).
- The professional development activities of teachers with technologies, within and outside of formal professional development programmes.
- Teachers' implementation of technologies in their classes.
- Meta-level reflections by teachers and researchers on the processes of professional development facilitated by the use of digital technologies.

These themes coexist, intersect and interact, possibly – but not necessarily – in sequence with each other. Moreover, this list does not aim at being exhaustive in that other dimensions could be considered (the affective dimension, the intercultural dimension etc.), but they are beyond the scope of our analysis.

This RF seeks to compare, combine and connect the most pertinent theoretical perspectives, in tune with an idea connecting theories (Prediger et al. 2008, see Figure 1) to describe and explain the whole process of mathematics teacher professional development with technologies.

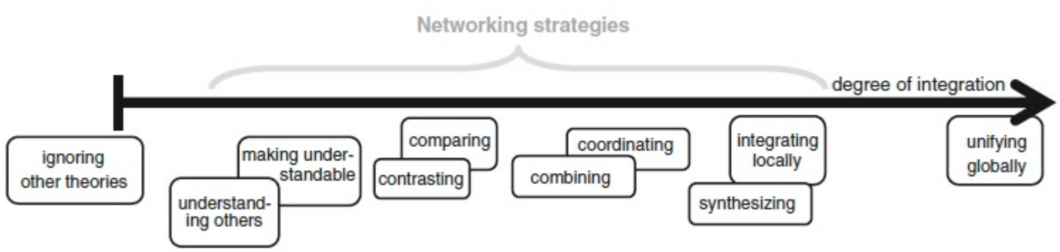

Figure 1 Networking strategies to connect theoretical approaches (Prediger et al. 2008, p. 170)

The idea is to explore what each theory can and cannot illuminate and to try to explain how they can work together. Thus, this contribution, the result of this co-working, develops around the afore-mentioned five themes. For each theme, we present examples from a variety of relevant studies from different contexts (country, professional development setting, type of technology, school phase, mathematical focus, pre- and in-service teachers, etc.), analysed according to different elements from the identified frameworks. This analysis is conducted with reference to specific sub-questions associated with each theme and makes it possible to highlight how the different theoretical ideas support the development of new understandings. The emphasis is on the usefulness of theories that enable both the temporal and personal aspects of teachers' trajectories to be described, with the teachers' voices as a central and essential element.

However, as the model of *Meta-Didactical Transposition* (*MDT*) may prove to be a useful tool for the analysis of different aspects involved in the whole process of teacher professional development, prior to focusing on each dimension and the corresponding examples taken from our studies, we briefly present an overview of *MDT* and highlight its main characteristics.

THE META-DIDACTICAL TRANSPOSITION AS A TRANSVERSAL LENS

The *MDT* model has been conceived to take into account the complexity arising from the intertwining of the processes involved during a teacher education program. It considers some main variables in the teacher education processes (community of teachers, community of researchers, role of the institutions) and accounts for the evolution of their mutual relationships. It includes a consideration of teachers' practices (both during professional development and in their activities in the classroom) and provides tools to analyse if and how teachers' knowledge and practice evolve during these processes. This evolution is observed as changes to and integrations of new teaching practices, mathematical technologies and research issues, both in the mathematics classrooms and within mathematics teachers' professional activities (programming didactical plans, designing tasks, planning assessment, etc.). Moreover, this evolution takes account of the teachers' relationships with institutions on the one hand and with the researchers' community on the other hand. Beginning with the assumption that institutions (i.e. national curricula, national assessment tools, the constraints of teachers' time and space, etc.) play an important role in the school context, the theoretical background for the *MDT* model is derived from Chevallard's *Anthropological Theory of Didactics* (Chevallard 1985, 1992). In particular the model refers to the notions of didactical transposition and praxeology. Chevallard defines didactical transposition as the transition from knowledge regarded as a tool to be put to use, to knowledge as something to be taught and learnt (Chevallard 1989). The notion of praxeology, which is the core of this theory, refers to the tasks that are to be performed and can be conceived as a quartet, constituted by two main blocks: (a) the technical-practical block, a task and a technique, that is the "know how" (which includes a family of similar problems to be studied, as well as the techniques available to solve them); and (b) the technological-theoretical block, constituted by the technology/technologies and the theory/theories that represent the argument that justifies or frames the technique for that task, that is the "knowledge" (García et al. 2006).

Since the aim of the *MDT* model is to frame and reflect on teacher education programs, the term "didactical" has been substituted with "meta-didactical" to stress that the processes under scrutiny are, in this case, the practices and the theoretical reflections developed within teacher education activities. In other words, in the case of teacher education programmes, fundamental issues related to the didactical transposition of knowledge are faced at a meta-level. Through the *MDT* model teacher education processes are analysed from a dynamic point of view, highlighting the interactions between the community of teachers involved in a professional development and the community of researchers who design and coach the activities. Initially, the two communities of researchers and teachers have their own praxeologies, associated to specific tasks. During the process of the *MDT*, as a result of the dialectical interactions between the communities, both the praxeologies of the community of researchers and the teachers' community change and sometimes evolve in a shared praxeology, which constitute the core element of the whole process.

Many factors have enabled us to identify the *MDT* model as a possible useful transversal lens that could act as a "binding agent" for the analysis of the examples we have chosen to discuss the five dimensions: (1) the stress on the role played by institutions and the constraints they impose; (2) the dynamic interplay and the interactions it allows to describe at different levels; (3) the focus on the different actors involved in these processes and on their mutual interactions; (4) the possibility it gives to highlight the evolution of teachers' and researchers' praxeologies over time through the notion of shared praxeology. Other aspects of the *MDT* model will be recalled and discussed in the analysis of specific examples, to include: the change of the status of some components of teachers' and researchers praxeologies from external to internal and vice versa; the brokering role played by teachers and researchers within the different communities; and the notion of double dialectic as a fundamental aspect typical of the processes aimed at fostering teachers and researchers' reflections and comparisons.

THE INSTITUTIONAL CONTEXT

There are two sub questions relating to this theme: To what extent can teachers develop individual agency in the face of institutional constraints, and what role can researchers play in this process? and How can researchers impact on the institutions in the planning of large scale professional development programs?

As Chevallard testifies (1987), the relationships between the institutions connected with the teaching system and society are most relevant,

> The teaching system is not a thing in one piece. It does not consist only of teachers and students, textbooks, homework assignments, and so forth. Like any social institution, it has to attend to the maintenance of its relations with society as a whole. Accordingly, a part of it will specialise in the overseeing of the relationship between the teaching system proper and

> its societal environment. This is a quite general requirement of social life, which no institution can elude. (Chevallard 1987 p. 2, our synthesised translation)

The *Anthropological Theory of Didactics* (ATD) focuses on the institutional dimension of mathematical knowledge and puts the activity of learning mathematics within the bulk of the human activities and of the social institutions (Chevallard 1999). Some examples of the relevant institutional variables are: the national curriculum; the ministry of education; national education programmes; national assessments; the textbooks; the schools and classes in which the implementation occurs; the communities of teachers of the same subject; and the communities of teachers involved in the same projects. These institutional components will vary in accordance with the national context in which they are situated.

The institutional context in teacher education activities is important in that it influences the choices made by stakeholders, researchers and trainers when a new programme of professional development is designed. By taking the institutional variables into account, it is possible to contextualise educational initiatives for teachers into the school setting, and indirectly, as a product of the professional development, the teachers' changed practices can have a positive impact on students and their mathematics competences. For example, many large-scale initiatives of this kind have happened in Italy over the few last years, and their impact is tangible from different points of view: the use of technologies by teachers and students; the increasing scores in international and national assessment of students; the diffusion of the new national curriculum; and so on (PISA 2012). In addition, the European Union is promoting lifelong education as strategic element for the development of countries, and in this context the institutional dimension is related not only to the educational one, but also to the political and social one. According to *ATD*, a mathematical object in school exists "since a person, or an institution acknowledges that *it* exists" (Chevallard 1992, p. 9). Consequently, we can also claim that a didactical object exists in a teacher education context since a researcher, or an institution, acknowledges that it exists. For example, an education programme based on teaching geometry through open problems with the support of a dynamic environment such as GeoGebra can be planned in a specific country taking into account: the national curriculum, the time teachers can spend in lifelong learning, the time they have available in school to introduce such types of tasks, the availability of classes, suites of computers, interactive whiteboards, and so on, namely all the variables that comprise the institutional dimension. In designing the activities for teachers, the researchers have to make these variables clear and make choices in relation to the specific objectives of the activities such that it is possible to impact upon teachers' learning. Alternatively, the teachers may not choose to take part in the training activity, or participate without consideration of the usefulness of professional development programme for their purposes at school. From the point of view of researchers who are involved in the design and implementation of the educational innovation by institutions as the Ministry of Education, or international organisation, or local institutions, it is very important to have not only the possibility to train teachers, but also to take the opportunity to disseminate key ideas from research within schools (contextualised through the curriculum, traditional methodologies, textbooks etc.). In this way, these are the mediating ideas between the institutional dimension related to teachers and that related to researchers (and in some cases the external institutional dimensions of private companies, the European Union, or others).

With reference to the model of *MDT*, we can say that the criteria on which the choice of the variables in the institutional dimension is based are part of the researchers' praxeologies. An example is a national project in Italy, *Piano Lauree Scientifiche* (PLS - Scientific Degree Plan), for which one sub-project of teacher education is "Problem solving with GeoGebra" (Robutti 2013). The organization of the programme began with an analysis of the new Italian national curriculum *Indicazioni Nazionali* (Ministero dell'istruzione dell'università e della ricerca 2010) in order to select the curriculum statements that could constitute the starting point for teacher professional development activities. The sections of the curriculum chosen for the design of tasks for teachers focus on both general aspects, related to the purposes of mathematical activities, and specific aspects, such as the role played by geometry, modelling, open problems, and the use of the technology in these domains.

According to the model of *MDT*, the research community has the *task* of selecting (with some *techniques*) the variables (geometrical concepts, use of software, and modelling) to focus on the educational programme within the institutional dimension represented by the national curriculum. This selection is carried out with reference to the aims of the project, which are part of the technological-theoretical part of the researchers' praxeologies. Other variables, coherent with the researchers' theoretical background, may be taken into account in the design of teachers' activity (e.g. mathematics laboratory, open problems, mathematics discussion). This is an example of what we have previously called a change in status of some components of teachers' and researchers' praxeologies from external to internal. Initially, these variables may be external to "ordinary" teachers' praxeologies. However, through the professional development programme, they become progressively internal, a result of the meeting of teachers' and researchers' praxeologies, as evidenced by several cases from within the *PLS* project. The problems connected to a change in a curriculum in countries where the schools have to follow national recommendation for its implementation are those related to the change: teachers have difficulties in change something in their praxeologies, and researchers can help them in doing it. *MDT* is a model to describe the process of development, giving insights into the elements that change in this process.

Chevallard poses the question of integration in anthropological terms that are the *viability of technological tools in the class* and stresses the importance of the teachers and the institutional contexts in which they act. Indeed, this viability is conditional upon by many aspects that have been considered within research on the integration of technical objects: the epistemological effects of the tools, the mathematical renewals which can result from them, etc. But this integration can be only partial a weakly viable if we forget the teachers' role. Chevallard explains the origin of the weak integration thus: one tends to retain only the knowledge ("*le savoir*") and the student's "*rapport au savoir*", forgetting that those cannot exist alone, in a didactic vacuum, without a functionally integrating didactic "intent", which is left, in practice, under the teachers' responsibility, however seconds are these aspects implicitly judged (Chevallard 1992, p6, our synthesised translation). Chevallard's *ATD*, by focusing on the institutional dimension of mathematical knowledge, obliges researchers who want to study teachers' practices in mathematics to situate this activity within social institutions.

Several examples of research applying this point of view can be found in Clark-Wilson, Robutti & Sinclair (Eds. 2014). For instance, in Haspekian (2014), teachers' difficulties in spreadsheet integration have been explained by the changes that the spreadsheets introduce within mathematical objects, techniques and representations. The research was concerning the domain of algebra, where the use of spreadsheet was planned by the observed teacher in order to help 12 years students to enter in algebra. But the changes introduced by the spreadsheets in the algebraic domain impacted upon the praxeologies that are usually viable in this domain in the French education institution for this grade. The table below gives a quick insight of the distance between the whole algebraic culture in the French secondary education and the algebraic world carried out by spreadsheets.

"Values" of algebra	In paper pencil environment	In spreadsheet
Objects	unknowns, equations	variable, formulae
Pragmatic potential	tool of resolution of problems (sometimes tool of proof)	tool of generalization
Process of resolution	"algorithmic" process, application of algebraic rules	arithmetical process of trial and refinement
Nature of solutions	exact solutions	exact or approached solutions

Table 1 Algebraic worlds

This instrumental distance introduced by the tool goes beyond Balacheff's *computer transposition* (1994) as it concerns all of the mathematical and didactical organisations that are usually viable in the classroom for the institution concerned.

Thus, the new praxeologies did not immediately, nor easily, fit with the institutional constraints that weigh on teachers' shoulders: national curriculum, inspection regimes, education programmes, national assessments, textbooks… that are all institutionally situated. By considering the whole institutional context, one understands better the difficulties of integrating spreadsheets for teaching and learning algebra.

THE DESIGN OF MATHEMATICS TEACHERS' PROFESSIONAL DEVELOPMENT

We reiterate our perspective that 'professional development' encompasses a wide range of individual and collaborative activities across a broad range of structured and informal opportunities, which are constrained by country-specific and cultural boundaries and expectations. Central to all of these activities lies the development of a teacher's mathematical, pedagogical and technological knowledge and practice. Consequently, the notion of an explicit 'design' implies that there has been some fore-thought. Whilst there have been some research studies that have sought to articulate the processes and outcomes of more informal professional development activities (see Clark-Wilson et al. 2014 for examples), here we will focus on professional development that has been constructed for the purpose of developing teachers classroom uses of technology.

The importance of design in the planning of both teacher professional development projects and specific related tasks for teachers (and their students) is pervasively recognized (de Geest et al. 2009; Even and Loewenberg Ball 2009). The research community has a prominent role in designing activities for teacher education, and this design of meta-didactical trajectories is the task of researchers involved in education programmes, while the teachers involved learn to design didactical trajectories for their classes. In this way, design can be described at two levels: of the teachers' activities and of the students' activities.

Design of teachers' activities may include also the design of students' activities and a team of researchers involved in this design may work with various methodologies, according to the cultural tradition of the country. In Italy, for example, the team is usually constituted by academic researchers and teacher-researchers who plan activities for teachers' programmes that include students' activities. In most of these programmes (i.e. *M@t.abel* or *Piano Lauree Scientifiche-PLS*) teachers are asked to experiment the proposed activities in their classes, during or after the training, in order to observe processes and discuss them in a final meeting of the research team and the teachers.

Design includes not only different types of tasks (i.e. open/closed problems, tutorial activity with technologies, etc.) and different types of lessons (lecture, workshop, working in groups). It also includes the design of initial questionnaires, interviews, materials used for the lessons, references to institutional aspects and logbooks to observe and record processes in the class. According to the paradigm of *MTD* the information acquired in the initial questionnaires/interviews supports researchers to identify the teachers' usual praxeologies when teaching mathematics with technologies.

In the following example, we present some data related to an initial questionnaire proposed to teachers involved in an educational programme in Italy in the national project *PLS*. This data may help the participant of the Research Forum to discuss and respond to the questions, "What is the role of the use of digital resources as a component of teachers' professional knowledge?", and, "How we can describe possible changes in their use by teachers, when they meet researchers in educational programmes?".

For example, in the programme design the team can prepare questions such as: Do you use different technologies in your class? What software do you choose? What kind of problems do you propose to the students in order to use technology to solve them? In which ways do you think technology can be useful for the learning process? These kind of questions make it possible not only to make inferences about the technology used by the teacher, but also on the teaching practices adopted and the teacher's ideas about the role of technology in learning processes, that is, the teacher's praxeology. For example, an older, experienced teacher of secondary school responded to the previous questions with these words: "Then, usually, for example in this class we have an IWB, so usually I do not prepare some special kind of things, but surely it is like having a projector and computer there, so for example it is quite normal that we use GeoGebra to explain, depending on the subject, but ... this definitely."

We can infer that the praxeology of this teacher implies her sole use GeoGebra by herself, to demonstrate something at the whiteboard, to pose a task and to solve it, to justify a procedure, without including students in the work on problem solving. By observing this teacher during the professional development programme, and then in the classroom with her students, researchers may collect data on her praxeologies and identify if and when there is some change in them, related to the use of technology. For example, researchers may highlight if a certain teaching practice far from her traditional way of working, at a certain point, become a consolidated praxis in her activity with students. The *MTD* model helps researchers in describing this passage as a modification in one or more components, which from external become internal, and mark a change in the evolution of teacher's praxeology, as a result of the meeting with the research team.

A second example, in this case, is the work done in the Comenius Project *EdUmatics* (Aldon et al. 2013b) - aimed at developing resources for mathematics teacher education in the field of integrating technology into mathematics teaching. The resources for professional development, are directed towards teachers, and include a range of tasks for school students, aimed at: giving an introduction to the use of technologies, using representations in static and dynamic way, making use of videos for teacher training, obtaining functions as models of phenomena and mathematical configurations. These themes offer a choice of different uses of technologies in teaching mathematics, depending on the motivation and the preliminary knowledge and skills. During the design of these resources, the research team met in order to share not only the tasks to prepare, but also the teaching practices to extent the tasks in the classes, and a develop materials related to didactical suggestions. The general aim was not only to give teachers didactical resources, but moreover to give ideas about some of the important research themes that have underpinned the design of the resources. This activity of the EdUmatics team can be described in *MDT* with the praxeologies of the research team, shared by the various countries groups involved in the project. Using the terminology of the *MDT*, these praxeologies are made of task-techniques (the design of activities and teaching practices, considering the institutional dimension of secondary school and the teaching practices and technologies to be enhanced); technologies-theories (all the reasons to implement such tasks, teaching practices and technologies, such as the theoretical references adopted by the research teams – in this case, for example, the multi-representation of mathematical objects in technological tools and multimodality as two sides of the same coin, the documentational approach and didactical incidents, the use of CAS in classes from a theoretical point of view, and instrumental orchestration). The EdUmatics project gave the research team the opportunity to work together and to learn each others, sharing praxeologies of research and of resources design. The collaboration during the EdUmatics project is an example of a co-production in which researchers and teachers brought in the design of resources their expertise and competencies, as co-producers (Kieran et al. 2013).

PROFESSIONAL DEVELOPMENT ACTIVITIES WITH TECHNOLOGIES

Digital technology encompasses technology as both a tool to experiment with representations of mathematical objects and a medium through which to find and communicate information (Hegedus and Moreno-Armella 2009). The context for the following example is a local professional development offer in which teacher trainers used a web-based platform. In the preceding year, eleven teams of teacher trainers volunteered to modify and augment their usual training sessions through a combination of on-line and face-to-face instruction. The main aim of the training session was to allow trainers to change from their usual in-service format to a blended learning system, as explained in Aldon et al (2013a). Researchers who participated in this training session evaluated its outcomes by investigating how some teachers implemented the ideas that had been presented. In this particular example, the subject of the training session was "the use of algorithms and programming to do mathematics". This subject is part of the French national curriculum for students in high schools (from 16 to 18 years old), especially for students following a scientific stream.

It is often interesting to analyse the failure of a program to achieve its objectives as a means of showing the importance of theoretical aspects. As it happens, the institutional context and the lack of shared praxeologies brought about difficulties in the professional development of the teachers. The training session was organized into four different phases. The first involved presentation (of trainees, of trainers, of the aim of the training session, of the programming languages). The second phase was a face-to-face session during which fundamental algorithms were presented and implemented on computers. At the same time, trainees started the design of lessons for the classroom. The third phase was conducted in “distance” mode with the aim being for teachers to implement lessons in their classes, to share observations and analysis, and to present further development of algorithms. The fifth and last phase was a face-to-face phase of discussion about the classroom implementation.

Teachers following the training session were volunteers, but chose to participate more because of the subject matter than for the hybrid modes of presentation. What is highlighted in this example is the difficulty of bringing these teachers to really take advantage of distant and asynchronous exchanges. It was clearly apparent that the modification of the institutional contract (responding to the didactical contract) was too great for teachers to alter their training habits. During the first phase, all trainees logged on to the web platform and participated in the presentation activity. During the face-to-face session, the trainees began the elaboration of mathematical courses including use of algorithms and programming for their own class context. However, and despite the efforts of the trainers to animate the forums, send relaunches, and offer new contents and challenging problems, the trainees did not concur with the organisation of the training session and did not keep up their participation in the course. An important aspect of the training session was for teachers to develop reflexive thinking on their professional behaviour relative to the use of computers in their mathematics courses.

Algorithms used in the teacher training session can be considered as praxeologies and the interesting thing is to compare the trainers' praxeologies and the trainees' praxeologies in order to understand why the training session did not lead to a shared praxeology. Let us take the example of the work on Graham's algorithm, which is a method of computing the convex hull of a finite set of points in the plane with time complexity O(n log n). The trainers’ praxeologies included the justification for the study of this particular algorithm by trainees - as a link between mathematical knowledge and algorithmic knowledge at the level of mathematics teachers’ knowledge. The implementation of such an algorithm in the classroom was not planned but the transposition to the classroom of the idea of linking mathematical problems and algorithmic solutions was seen by trainers as a consequence of this task. At the same time, trainees considered this task as an application of sorting algorithms without possible applications in the classroom. The lack of discussion in the third phase of the session meant that the two praxeologies remained separate without ever becoming a shared praxeology. The consequence of the misunderstanding of the institutional contract was a rupture in the dynamic of the *MDT*, despite the mediation by the researchers.

A second example illustrates the situation where a teacher sought out informal professional development opportunities through professional networks and participation in research projects, rather than through working with teacher trainers. Again, however, the role of institutional contexts is evident. This teacher came to integrate digital technologies into his practice as a way of helping his students (secondary school age) to access the curriculum and succeed in learning mathematics. This summary of his development draws on data from his participation in several research projects between 2001 and 2010. Until the 1990s he would have described himself as a traditional teacher who tried to explain mathematical concepts to students as clearly as possible. He expected students to copy what he did and to demonstrate their recall on tests. However, he was confronted with the reality that most students did not understand what he was teaching because, only a few weeks after passing the test, they seemed to have forgotten everything they had learned. Rather than blaming the students for their apparent inability to learn, he returned to the university where he had completed his initial teacher education to look for new ideas in mathematics teaching through discussion with academic researchers and reading current literature. In subsequent years he volunteered to participate in research projects investigating the role of digital technologies in mathematics teaching and learning. In this way he created his own pathway of development in response to the pedagogical problems he wanted to solve.

As a result of his professional reading, this teacher became influenced by the work of Paul Ernest on constructivism, and he began to reform the curriculum and teaching approaches in his school along constructivist lines. He had previously attended professional development workshops on the use of graphics calculators (which were, at the time, a new form of technology being introduced into secondary school mathematics). Initially he saw graphics calculators and other technology as being “interesting but not essential.” However, when his teaching philosophy changed he realised that technology was a way of helping students to access concepts that would otherwise be beyond their understanding. In addition, at this time the use of graphics calculators and computers had been made mandatory in senior secondary mathematics curricula. As Head of the school’s Mathematics Department he developed a new junior secondary curriculum incorporating manipulatives and digital technologies, with a blend of student-centred small group work followed by whole class teacher-led discussion. Other teachers were initially resistant to this new approach because it demanded more pedagogical flexibility than they were accustomed to using. However, they quickly became convinced of the benefits when they saw that students whom they thought incapable of learning could succeed when given appropriate tasks – many of which were technology-enriched – in contexts that encouraged dialogue and experimentation.

In this school the teacher began to develop a new identity by participating in new professional practices – those centred on both his own learning and his students' learning (Wenger 1998). The institutional context was an important influence on his developmental trajectory, offering potential enablers and hindrances. For example, the university offered access to academic experts and research literature that "seeded" the teacher's thinking about constructivist pedagogies. Without this, the graphics calculator workshops in which he had participated would not have been seen as useful. His school context could have hindered his development due to lack of resources and teacher resistance, if not for the support he received from the Principal in initiating change. The development of new senior secondary mathematics curricula that mandated technology use gave the teacher another argument for introducing graphics calculators and computer applications in the junior secondary years. This example shows that institutions can have many, sometimes conflicting, and influences on teacher professional development. These influences are not static; instead, they interact with a teacher's search for professional development opportunities that align with his/her goals and problem solving needs. Conceiving of teacher development as identity formation makes it possible to trace out the dynamic, temporal dimension of professional learning.

This section has presented two brief examples of the evolving role of teachers in terms of their professional development activities with technologies. The examples have illustrated partial successes and failures, using different theoretical lenses. Together, however, they allow us to observe teacher change (or not) (i.e., the first key question posed by this Research Forum) and the role of institutions such as school curricula, professional development regimes, and societal expectations in supporting or hindering change (the second question for this RF). A related question that could be posed is whether the extent to which a teacher has mastered a mathematical digital tool supports them to transform the tool into a didactical professional instrument (see below). The relevance of this question is less evident in the second example than in the first, where teachers' mastery of virtual communication technologies came into play. Nevertheless, as both examples suggest, mastery of the digital tool is but one of many factors that may influence teachers' use of technologies in their classes.

As we saw in the earlier section, which considered institutional contexts, some research has stressed the importance of taking into account of the instrumental distance generated by the tool between the different praxeologies that are viable in the different environments, the new technological environment and the usual paper-pencil one. The necessary work to relate these praxeologies is part of the teacher's professional instrumental genesis. Using the frame of the Instrumental Approach, Haspekian (2011) uses Rabardel's notion of instrumental genesis and distinguishes personal from professional genesis by distinguishing two different instruments for the teacher. From a given artefact, the personal instrumental genesis leads to the construction and appropriation of an instrument for mathematical activity. From, or along with, the previous instrument, the professional instrumental genesis leads to the construction and the appropriation of a didactical instrument for mathematics teaching activity. Indeed, the teacher has to turn the digital tool into a didactical tool in order to serve her learning objectives. This task is non-trivial, even if teachers have been made aware of the digital tool's didactic potentialities, and even if didactic work in terms of situations has been already done. The situation becomes more complex when the digital tool is a non-educational one, encompassing a personal genesis and a professional genesis on the teacher's part. The relationship between personal and professional geneses is accentuated in the case of technologies that are not initially made for mathematics education and are, such as spreadsheets, imported into classrooms to teach mathematics. The case studied in (Haspekian 2014), shows that teachers' personal and professional instrumental genesis cannot be independent and that this double instrumental genesis of the teacher can also interfere with the students' development.

TEACHERS' IMPLEMENTATION OF TECHNOLOGIES IN THEIR CLASSES

There is a substantial body of research on how particular teachers in particular settings have integrated particular technologies within their classroom settings (Hoyles and Lagrange 2009). As has been previously stated, although the earlier studies used this context to research the mathematical outcomes from the students' perspectives, more recent studies have focused on the process of the teachers' development of the knowledge and classroom practices over time. This has led to a number of global and local theories that served both to explain particular classroom outcomes and to inform the development of professional development programmes and ongoing support through professional learning communities. This section focuses on three evolving approaches - the Technological Pedagogical Content Knowledge (*TPACK*), the Instrumental Approach (*IA*) and *MDT*. The question of technology is therefore tackled using concepts emanating either from an ergonomic approach (instrumental genesis as in *TPACK*), or from an anthropological approach (as didactic transposition in *MDT*), or from both (as in the *IA*). The section is exemplified by particular examples of individual teachers' trajectories. These individual stories provide insight into how the particular features and functionalities of the different digital mathematical tools impact upon teachers' motivation and confidence to integrate them into classroom teaching and how they respond to the challenges of task design involving mathematical digital technologies. They illustrate the use of different theories

TPACK (Mishra and Koehler 2006; Koehler and Mishra 2009), similarly, but complementary to Pedagogical Technology Knowledge (*PTK*) (Thomas and Hong 2005; Hong and Thomas 2006) has demonstrated merit in analysing factors related to the challenges that teachers face in using digital technology, and provides an indication of teacher readiness for implementation of technology use. A critical review of the TPACK frame, including an analysis of its affordances and constraints can be found in Graham (2011). While *TPACK* takes a more generic approach, *PTK* is mathematics focused, recognising that mathematics has its own important nuances of content knowledge, as exemplified in Ball and Bass's framework of mathematical knowledge for teaching (MKT - Hill and Ball 2004). In turn, it places an emphasis on the epistemic value of the technology, how it can be used to produce knowledge of the (mathematical) object under study (Artigue 2002; Heid et al. 2013). Both frameworks build on Shulman's pedagogical content knowledge by adding aspects of (digital) technology knowledge (PCK - Shulman 1986). Here, *TPACK* articulates each of *PCK*, technological pedagogical knowledge (*TPK*) and technological content knowledge (*TCK*) and the relationships between them. In the framework, *TCK* involves an understanding of the manner in which technology and content influence and constrain each another, while *TPK* is an understanding of how teaching and learning can change when specific technologies are used in particular ways (Koehler and Mishra 2009). The definition of technology knowledge (*TK*) used in *TPACK* to form the constructs of *TCK* and *TPK* is close to that of Fluency of Information Technology (FITness), as proposed by the Committee of Information Technology Literacy of the National Research Council. (Koehler and Mishra 2009, p. 64). Also, *PTK* includes the crucial element of the personal orientations of the teacher who is using the technology and their role in influencing goal setting and decision-making. Hence, it suggests teachers need to understand information technology broadly enough to apply it productively at work and in their everyday lives, to recognise when information technology can assist or impede the achievement of a goal, and continually to adapt to changes in information technology. In contrast, *PTK* highlights the principles, conventions, and techniques required to teach mathematics through the technology. This includes the need to be a proficient user of the technology, but more importantly, to understand the principles and techniques required to build and manage didactical situations incorporating it and enable mathematical learning through the technology. Thus, *PTK* employs the theoretical base of instrumental genesis, with its explanation of how tools are converted into didactic instruments, while *TPACK* relates to "knowledge of the existence, components and capabilities of various technologies as they are used in teaching and learning settings, and conversely, knowing how teaching might change as a result of using particular technologies." (Mishra and Koehler 2006 p. 1028). However, while there are differences in the frameworks it is clear that both provide useful conceptual lenses for analysing classroom practice, and should be viewed as complementary rather than competitive.

The *PTK* and *TPACK* frameworks suggest that the ability of a teacher to employ digital technology to construct and use tasks with epistemic value requires sound technological, pedagogical and content knowledge, along with positive orientations towards learning and teaching with technology, good *MKT* (Hill and Ball 2004) and sound instrumental genesis. Thus, a teacher's perspective on the technology, their familiarity with it as a teaching tool, and their understanding of the mathematics and how to teach it are all crucial factors. A teacher with strong technological, pedagogical and content knowledge can understand the principles and techniques required to build didactical situations incorporating digital technology, comprising tasks that enable mathematical learning to emerge, mediated by the technology. We exemplify such knowledge here in the case of two secondary mathematics teachers.

The first case, reported fully in Thomas and Hong (2013), describes a teacher who had moved forward in the use of digital technology. In spite of over six years' experience of using graphic calculators (GC) in her teaching she admitted "Sometimes it's hard to see how to use it effectively so I don't use it as continuously as I should." Her confidence was, however, at a level where she had "… done some exploratory graphs lessons where students get more freedom to input functions and observe the plots." Thus, she was happy to loosen control of the students and let them explore the GC and help one another: "Students learn a lot by their own exploration…In past lessons I have never had a student get lost while using a graphics calculator. Sometimes friends around will assist someone."

She expressed a desire for her students to appreciate the challenge of the depth of mathematics: "The success for me as a teacher is when they want to learn more and students show a joy either in what they are doing or in challenging themselves and their teacher with more deeper or self-posed mathematical problems." She was convinced that the technology could be used to challenge and motivate students in this way "The calculator puts a radiant light in the class… With a graphics calculator lesson no one notices the time and no one packed up." An orientation, a belief, crucial to her pedagogical technology knowledge was related to the complementary roles of by hand and technology approaches. This was revealed through her comment that "Today we find a lot of maths does not need underlying understanding…I feel as teachers what we need to really be aware of is what the basics are that students must know manually… when we sit down to work with graphics calculators we need to consider carefully what still should be understood manually."

One of her lessons, with a class of 17 year-old students, considered families of functions with the aim of exploring exponential and hyperbolic graphs and noting some of their features, "we're going to utilise the calculator to show that main graph and then we're going to go through families of $y = 2^x$". Her pedagogical technology knowledge enabled her to direct them to link a second representation, "Another feature of the calculator I want you to be aware of..[pause] you've got also a list of x and y values already done for you in a table." Her instrumental genesis was such that she had moved away from giving explicit key press instructions, instead declaring "I want you to put these functions in and graph them and see what's going on." and "You can change the window if you want to see more detail, and if you want to see where it cuts the x-axis, you can use the "trace" function." A copy of her whiteboard working can be seen in Figure 2.

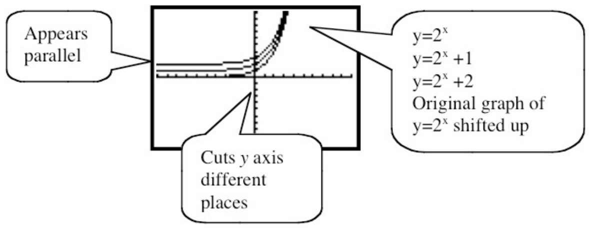

Figure 2 The teacher's whiteboard working (Thomas and Hong 2013)

She was also moving towards an investigative mode of teaching "if you're not sure where the intercepts are, you can use the "trace" key, remember, and I want you to observe what is happening", encouraging students to use the GC in a predictive manner, to investigate a different family.

We want to do some predictions… Looking at the screen try to predict where 3 × 2x will go then press "y =…" and see if it went where you expected it to go. You may get a shock… Can you predict where "y = 4 × 2x" will be? Now you learned from that, so can you predict where it'll lie. The gap between them gets smaller. If you're interested put in "y = 100 × 2x". Does it go where you expect?

The epistemic value of the teaching was noticeable since mathematical concepts were a focus of attention. For example, she linked 2×2^x with 2^{x+1} and during an examination of the family of equations $y = 2^x$, $y = 2^{x+1}$, $y = 2^{x+2}$, said of y=2^{x+1} "We expect this to shift 1 unit to the left [compared with 2^x]. Did it?" In this way she encouraged versatile thinking by linking with previous knowledge of translations of graphs parallel to the x-axis, and reinforced this with the comment that "With this family, when you look at the graph can you see that the distance between them stays the same because it's sliding along 1 unit at a time. The whole graph shifts along 1 unit at a time." This relationship between the functions is not so easily seen by students from the graphs and hence she linked to the algebraic expression and the foundation of a previously learned mathematical concept. In addition, there was a discussion of the relationship between the graphs in the family of $y = 2^x + k$, and the relative sizes of 2^x and k.

…as the exponential value gets larger, because we're adding a constant term that is quite small, it lands up becoming almost negligible. So, when…all they're differing by is the constant part, you'll find that they appear to come together. Do they actually equal the same values ever? Do they ever meet at a point? No, because of the difference by a constant, but because of the scaling we have, they appear to merge.

In summary, she had demonstrated good technological, pedagogical and content knowledge. Her technological knowledge had reached the point where she showed strong instrumentation and instrumentalisation of the technological tool. Thus, she was able to use the affordances of the technology (within acceptable constraints) to provide an epistemic focus on mathematical constructs. This included the idea of testing concepts against definitions, a strong emphasis on the crucial process of generalisation (Mason et al. 2005), and the use of student investigation to form and test conjectures. In addition, she had a high level of confidence in using the technology to teach mathematics and positive orientations, including a strong belief in the value of technology as a tool to learn mathematics.

Teachers' implementation of technology in their classes can also be studied at the local level of instrumental geneses using the Instrumental approach (IA). An example of interference of the teacher's double instrumental genesis and students' ones is given in the case of spreadsheet already mentioned in the sections before. In this study of the teacher integrating spreadsheet for algebraic learning, these relationships are constrained by:

- The mathematical knowledge aimed at (statistics, algebra, etc.)
- Pupils' instrumentation (that is how to make pupils work mathematics *through* spreadsheet, encompassing instrumental and mathematical knowledge, for example: frequency, dependence through the change of the value in the cell)
- Pupils' instrumentalisation (that is which functionalities, schemes of use are aimed at? For example: relative references, recopy, incrementation with the copy, but not absolute references, $ sign and its different behaviour in the copy)

Managing all these constraints at once is not easy as a spreadsheet is not automatically a didactical instrument, the case studied here shows that such an instrument is only progressively built along a complex professional-oriented genesis and that the professional and the personal geneses interfered one on the other.

For example, in preparing the task for pupils, the teacher modified her spreadsheet file 3 times! (see Figure 2) In its 1st version, the formula calculating the frequency (in B7) was: **=B6/50*100**. This formula, if copied along line 7 is convenient for Qa) but not anymore for Q b) (The formula refers to the value 50 for the total. If one changes the value of any cell, then the total will change and the form becomes wrong)

The day before the lesson, the teacher realised the mistake and changed the formula into: **=B6/F6*100**. She confided she did not feel yet totally comfortable with spreadsheet. If her own instrumental genesis with spreadsheet-as-a *mathematical* instrument probably plays a role here, we also see that the key point of the problem comes from the spreadsheet-as-a *didactic-oriented* instrument. It is the didactical aim (showing the mathematical dependency between the numbers and the frequencies) that led the teacher to add Qb) and make pupils change the number in C6, which turned wrong the formula. She did not realise it when she built first her formula. At that moment, the personal instrument stands at the front of the scene, and covers up the professional and its didactical aims (the Qb.).

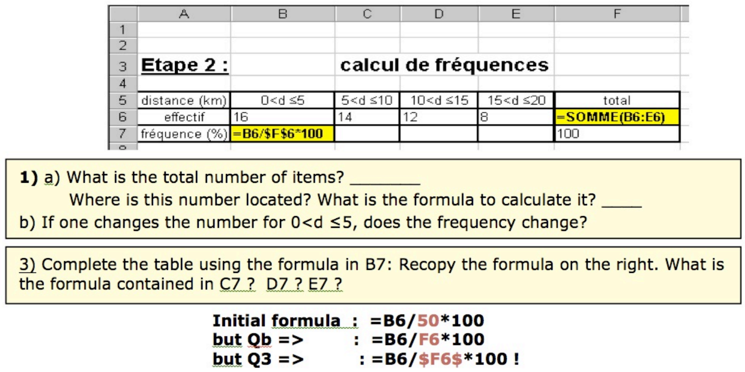

Figure 3 The teacher's spreadsheet and accompanying worksheet (Haspekian 2011)

In this example, the teacher's spreadsheet session has been disturbed because the teacher wanted to avoid mentioning the $ sign to the pupils, but it came out during the session! Facing pupils' questions, she was compelled to explain but she just said that it is not important to write it in paper-pencil. This link with the paper-pencil work is a strong preoccupation for teachers and is precisely linked to the instrumental distance generated by the tool evoked within Section 1.

The final example of implementation of technologies in the classroom is analysed in term of *MDT* in the context of the European project EdUmatics (Aldon et al. 2013b). In France, a high school teacher (called Jean in the following) worked in collaboration with the French Institute of Education (ENS de Lyon) and the Italian team in Turin (made of researchers and teachers). The purpose of the project was to develop professional development activities for teachers of mathematics in Europe. It was therefore necessary, to transform classroom situations into training situations. Jean's role was to adapt and analyse a mathematical task for students that had been created by the colleagues from Turin. Jean said, "All of this work led me to reflect on professional actions from a training perspective. This reflection is of course beneficial for my own training! […] The preparation work was often meticulous, observing the influence of gestures or seemingly innocuous words in the course of a session, which helped me to improve my classroom management. This experience has allowed me to build some of my pedagogical beliefs, including the conclusion that the exchange and mutual building of knowledge with students may be preferable to a lecture" (EducTice-Info 2 2012). In this case, and with reference to the *MDT* framework, the implementation of technologies within the classroom is the result of the evolution of praxeologies taking into account both the point of view of the research and the point of view of teacher professional development. The task, designed in another institutional context, find its justification in the French institutional context because of the *a priori* analysis leading to a shared praxeology by researchers and teachers.

This section has presented detailed examples of analyses with three different lenses of teachers' implementation of technologies in their classes. These different lenses provide tools to analyse on one part the evolution of practices of mathematics teachers with technologies; on the other part the impact of this use of digital tools upon teacher's professional development, which were two of our key questions in this Research Forum.

The lenses are different but complementary For instance the evolutions of the two teachers in the first example (TPACK) can be complementary tackled with the tools of the Instrumental approach. In the case of the first teacher, it has been said that her instrumental genesis moved from giving explicit key press instructions to a more exploratory mode ("put functions in and graph them and see what's going on"). In fact, the genesis at stake here is that of the *professional instrumental genesis* because it is the GC as a *didactic* tool that is being progressively built here, not the GC as a personal tool for the teacher (calculating or plotting). This evolution implies constitution of schemes of instrumented action as the one described in this example 'moving from key press instructions to open questions; moving "towards an investigative mode of teaching", "encouraging students to use the GC in a predictive manner"…).

Last but not least, the implementation of technology in classroom also poses the question of its link with educational programs for teachers. Thus, related questions to this section could be: To what extent can teachers develop individual agency in the face of institutional constraints, and what role can researchers play in this process? and How can researchers impact on the institutions in the planning of large scale professional development programs?

META-LEVEL REFLECTIONS BY TEACHERS AND RESEARCHERS IN THE PROCESS OF PROFESSIONAL DEVELOPMENT FACILITATED BY THE USE OF DIGITAL TECHNOLOGIES

Teachers involved in the different activities which characterise the process of professional development, according to the different roles they could play (as teacher-researchers, or trainers, or ordinary teachers), may reflect on their activity and evolve in their praxeologies over time, if motivated to the importance of that, and if helped by researchers.

The meta-level reflections that teachers and researchers can carry out are part of the professional development as a whole process. Moreover, there are recent studies that actually highlight that involving teachers in reflective practices where classroom dynamics are object of a careful scrutiny, enables the teachers' deep beliefs emerge, so that the reconstruction of a new identity for the teacher becomes possible (See for instance Goos 2013; Jaworski 2012).

The notion of *double dialectic*, a component of the *MDT* model (Aldon et al. 2013a; Arzarello et al. 2014), could enable to introduce and analyze this dimension. This construct has been conceived to highlight a typical feature which characterizes those teacher education programs that are based on the study of the teachers' practice: the engendering of dynamics which enable the teachers develop an awareness about their role during classroom activities and also possible gaps between their knowledge and beliefs and their classroom actions.

The double dialectic encapsulates two interrelated processes: (1) a *first dialectic*, which is at the *didactic level* in the classroom, between the personal meanings that students attach to a didactic situation to which they are exposed and its scientific, shared sense; (2) a *second dialectic*, which is at the *meta-didactic level*, between the interpretation that the teachers give to the first dialectic according to their praxeologies and the meaning that the first dialectic has according to the community of researchers, which results from researcher praxeologies. It is through this double dialectic that, thanks to the constitution of a shared praxeology, a significant evolution of teacher professional competences could be fostered. The use of digital technologies as tools to promote teachers' reflections on the educational processes in which they are involved (Hegedus and Moreno-Armella 2009) further facilitate the engendering of this double dialectic.

The first example is therefore related to the use of technologies as tools for teachers to communicate and interact with researchers and mentors. It proposes possible activities that could activate this double-level process: those connected to the *Multi-commented transcripts* methodology, developed within the *ArAl Project*. The *ArAl Project* (ArAl is an acronym for "Arithmetic and Algebra") is aimed at proposing a linguistic and constructive approach to early algebra starting from primary school or even kindergarten and is also meant to constitute an integrated teacher education program (Malara and Navarra 2003; Cusi et al. 2010). The *Multi-commented transcripts* are the results of a complex activity of critical analysis of the transcripts of audio-recordings of classroom processes and associated reflections developed by groups of teachers and researchers involved in the same teaching experiment within the *ArAl Project*. The teachers who experiment the project activities in their classes send the transcripts, together with their own comments and reflections, to mentors-researchers, who make their own comments and send them back to the authors, to other teachers involved in similar activities, and sometimes to other researchers. Often, both teachers and researchers make further interventions in this cycle, commenting on comments or inserting new ones. This process, which is carried out through email exchanges, is characterized by a sort of choral web participation because of the intensive exchanges via e-mail, which contribute to the fruitfulness of the reflections emerging from the different comments. These activities have been conceived, in a perspective of lifelong learning, starting from the hypothesis that involving teachers in the critical-reflective study of teaching-learning processes, to be developed within communities of inquiry (Jaworski 2003), could enable their development of awareness about the "subtle sensitivities" (Mason 1998, 2008) that could guide their future choices and determine their effective action in the classroom. Through the Multi-commented transcripts teachers have the possibility to become aware of: (1) the contrast/interaction between the personal sense their students attribute to class activities and the institutional meaning of both the same activities and the mathematical concepts involved (*first-level dialectic*); (2) the possible different interpretations, given by teachers and researchers, of the dynamics activated during class activities (*second-level dialectic*). The tension developed as a result of this double-level dialectic fosters the development of new teachers' praxeologies, related both to the roles they should activate in their classrooms and to the ways of pursuing their professional development.

Digital technologies have enabled an evolution of the Multi-commented transcripts. The initial activation of the ArAl Project official website (www.aralweb.unimore.it) and the recent activation of a work-in-progress blog (http://progettoaral.wordpress.com) have, indeed, created "virtual places" where teachers can find clarification and further materials on mathematical, linguistic, psychological, socio-pedagogical, and methodological-didactical issues and also prototypes of didactical sequences aimed at giving them a stimulus for their own elaboration of teaching processes. The blog, in particular, is a source of information for all the teachers who are interested in classroom innovation and, therefore, a "place" where the dialogical comparison typical of the ArAl Project can further develop.

The evolution of the Multi-commented transcripts, the *Web Multi-commented transcripts*, are interactive PDF-files conceived as learning tools to enable the reader to develop an in-depth analysis of the presented activities, through web-links to both the website and the blog that highlight: (a) specific theoretical terms used in the teachers and researchers' comments; (b) contents related to theoretical, methodological and disciplinary aspects; (c) some FAQ, possible answers aimed at clarifying important aspects often highlighted by many teachers involved in the project through their comments.

The methodology of Multi-commented transcripts have therefore evolved from professional development tools for the teacher's own reflections to tools to be shared within the whole community of teachers and researchers, specifically conceived to be used as formative web objects to mediate theoretical aspects and classroom practice. The Web Multi-commented transcripts are therefore examples of how technologies as communication infrastructures can impact and strengthen specific tools conceived for teacher education, enabling also to highlight the role played by teachers as protagonists of their own professional development, through the interaction between their "voices" and researchers' voices in educational programmes.

A second example comes from the *Cornerstone Maths* Project (Hoyles et al. 2013; Clark-Wilson et al. accepted), which has developed a *curriculum activity system* (Vahey et al. 2013), comprising curriculum teaching units with integrated dynamic software and accompanying professional development and community support for selected mathematical topics in lower secondary mathematics education. This national project is researching the design and impact of the introduction of dynamic mathematical technologies at scale, with over 230 teachers and 6000 students currently involved. In this case, the participating teachers are introduced to the online community during face-to-face professional development and encouraged to continue to use the fora to discuss ongoing aspects of their developing classroom practices, share their lesson adaptations and reflect upon their students' learning outcomes. Just over two thirds of the teachers who have completed their teaching of the first Cornerstone Maths unit of work on linear functions (n=78) reported that they had made use of the Forum beyond the initial face-to-face PD. They cited the following uses for the community: to keep up to date with the project news (n=28); to read questions and comments by the community (n=45); to post questions or comments to the community (n=11); to access electronic copies of the pupil workbook and teacher guide (n=17); upload resources they had created to share with the community (n=3) and to download resources created by others (n=3). Although this is early data from the project, and further analysis of the qualitative data contained within the community's written exchanges will reveal the nature of its role within teachers' professional learning trajectories, the results do justify the creation of such online communities in both establishing a professional community and in enabling ongoing professional discourse for the project's participants.

CONCLUSIONS AND PERSPECTIVES ON THE FUTURE RESEARCH

Teaching mathematics with digital technologies is a challenge that teachers in different countries have faced to differing degrees both individually and as practitioner communities. In this paper we have described how research can approach this theme and describe it from different perspectives, which are more or less integrated. Our approach has been to consider the different aspects of teaching within both professional development – to include the design of activities by researchers and/or teachers that are mindful of the institutional considerations – and within the classroom. These aspects are contextualised within the process, which often pass through face-to-face or distance learning phases, through to the post-implementation reflections on those activities. In this way, the term professional development is intended in a wide sense, and is seen as a process involving many actors (researchers, trainers, teacher-researchers, teachers as learners and as teachers in their classes, mentors). In this process, all the actors may change their ideas and approaches to the use of technologies, and so their praxeologies may evolve, thanks to their dialogical interaction with the other actors.

In this paper we have highlighted the elements that signal this evolution, both from the point of view of research and of teaching practices. The selected frameworks were used in a co-working approach to describe the variables within the professional development process concerning teaching mathematics with technologies. The various frameworks highlighted the description of the professional development in different ways, taking into account the activity of teachers, the institutional aspects and the relationships within professional development settings. For example, *MTD* accounted for the dynamic aspects of PD and allowed us to consider both the perspectives of both teachers and of researchers (or teacher trainers) in a joint action. By contrast, *PTK* revealed a picture to help us understand teachers' classroom practices and the relationships between teachers orientations and the possible use of technology. Instrumental genesis, combined with an analysis of pedagogical and technological knowledge, provide tools that give a clear description of mathematical constructs with technology and enable us to tackle the complexities of the dynamic process of the related instrumental geneses (personal/professional or students'/ teachers' ones).

Such frameworks give tools to highlight the important evolution of teachers' professional learning about using technology in mathematics lessons and enable us to capture the importance of the didactical and pedagogical aspects, which are linked to the constraints and potential of technology, They also point the importance of the role of institutions. For example, the *MTD* model and the Instrumental Approach, are both connect to Chevallard's ATD through the notions of institution, didactic transposition, or praxeologies. Thus, the variety of theories mentioned here share the common point of having this "sensitivity" to contextual and individual factors that may account for the evolving role of the teacher in technology-enriched mathematics teaching. Moreover, teachers' personal beliefs about what represents a good teaching are situated in specific institutional cultures (Goos 2014).

Examples that have been developed in the text also show, if needed, the importance of the phase of designing a lesson when using technology as well as the accompanying role of research. Researchers and teachers, when working together, provide examples of possible evolutions of teaching practices with technology that takes on a share of professional development advancement, particularly by developing meta-levels of reflections on both the educational processes and the results of these processes.

Evolution of practices and innovation have been observed and described through theoretical frameworks that allow understanding the use of digital technological tools for teachers and the associated professional development. A large scale mathematics teachers' professional development and its institutional implications need to be developed in future researches, leaning on the first results described in this paper. Particularly, evolution of practices and innovation within mathematics teaching may be accompanied by strong researches giving evidence that would be usable in the designing of PD sessions.

Finally, we acknowledge that it was not possible to be exhaustive in our coverage of research perspectives and approaches that exist for analysing technology mediated teaching. For example, the *Documentational Approach* (Gueudet and Trouche 2009) and *Semiotic Mediation* (Bartolini Bussi and Mariotti 2008). However we expect the Research Forum to provide the opportunity to discuss and debate other related theories.

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